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GEOMETRIC SHELL STABILITY THEORY

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by

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INTRODUCTION

Shell is called the elastic body, limited by two surfaces, the distance between which (thickness of shell) is small in comparison with the remaining size/dimensions of this body. The surface, which separates the thickness of shell in half, is called median surface. When they speak about the form of shell, they usually bear in mind the form of its median surface. As elastic body, shell under the action of the applied to it load undergoes strain. In this case, in its material, appear the voltage/stresses. The basic task of the theory of shells consists of the determination of strains and voltage/stresses, caused by the effective on shell load. The voltage/stresses in the material of shell are located simply, if is known the deformation of shell. Therefore, it is possible to count that the task indicated is reduced to the determination of the deformations of shell.

Let us visualize that shell F is deformed, taking form F'. During this deformation the internal effort/forces, which appear in the material of shell, produce certain work. This work is called strain energy.

$$\sum g_{ij}\, du^i du^j, \quad \sum h_{ij}\, du^i du^j, \quad i, j = 1, 2.$$

- first and second quadratic forms of initial surface, and

$$\sum g'_{ij}\, du^i du^j, \quad \sum h'_{ij}\, du^i du^j$$

- corresponding shapes of surface F'. Let us designate through ε_1 and ε_2 the outer limits of the ratio/relation

$$\frac{\sum (g'_{ij} - g_{ij})\, du^i du^j}{\sum g_{ij}\, du^i du^j},$$

and through $\varkappa_1$ and $\varkappa_2$ - outer limits of the ratio/relation

$$\frac{\sum (h'_{ij} - h_{ij})\, du^i\, du^j}{\sum g_{ij}\, du^i du^j}.$$

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Then the strain energy $U_{F'}$, connected with the transition of shell of form of F in F', it is calculated from the formula

$$U_{F'} = \int_F \int \frac{E\delta^3}{24(1-\nu^2)} (\varkappa_1^2 + \varkappa_2^2 + 2\nu\varkappa_1\varkappa_2)\, d\sigma + \\ + \int_F \int \frac{E\delta}{2(1-\nu^2)} (\varepsilon_1^2 + \varepsilon_2^2 + 2\nu\varepsilon_1\varepsilon_2)\, d\sigma,$$

where E and ν - the elastic constants of the material of shell, δ - its thickness, and integration is fulfilled by surface area F.

If to shell is applied certain load q, then during the deformation of shell of form of F in F' this load produces certain work. Let us designate it through $A_{F'}(q)$.

One of the methods of solving the basic task of the theory of

shells is based on the following variation principle.

Under the action of the assigned/prescribed load q, among all possible forms F', that satisfy the conditions of attachment, the shell takes such form, on which the functional

$$W = U_{F'} - A_{F'}(q)$$

is stationary, that is has equal to zero variations.

It is easy to present sowing the difficulties which appear during the solution of the variational problem indicated. If we designate through w_1, w_2, w_3 the components of displacement of the points of surface F during its deformation in F', then integrand for strain energy $U_{F'}$ is sufficiently complicatedly the expression, which contains functions w_i and their derivatives of the first and second order. Therefore the solution of this task even in the simplest cases is virtually impracticable.

If we previously to assume that the form of the deformed shell F' is close to initial (F), then functional W logically is simplified and is led to quadratic. The corresponding system of equations of Euler for functions w_i, which realize the extremum of functional W, will be linear. The solution of basic task in this simplest case composes the object/subject of the linear theory of shells.

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Let shell F be located under the action of certain load q. If load is small, then the elastic state of shell among the forms, close to F, it is determined unambiguously. We will increase load q. Then can begin this torque/moment, when by the condition of nearness indicated the elastic state of shell is not unambiguously determined. Specifically, together with the basic form of the elastic equilibrium of the shell for which the deformed surface of shell remains close to initial form (F), also, during further increase in the load there are other forms which are developed virtually without an additional increase in the effective load, moreover this development is accompanied by considerable changes in exterior form of shell. The smallest load with which occurs the indicated ambiguity of elastic states, is called upper critical load, and transition to the nonbasic/minority forms of elastic states - by a loss of stability of shell.

The elastic states of shell, which appear as a result of loss of stability, we conditionally call supercritical. Determination and investigation of these states is substantially nonlinear task. Solving it, usually are assigned by the character of the sagging/deflections of shell, reducing thus the variational task for functional W to task to extremum for functions from the parameters,

which characterize deformation. In this case, the result significantly depends on how are successfully selected the functions, which assign deformation. The proposed by us method of the study of the supercritical elastic states of shells is based in its essential part during geometric considerations and it consists in general terms of following.

First of all, we proceed from assumption about the fact that the supercritical deformation of shell is in essence geometric bending. This is not difficult to base. Really/actually, running structural/design materials - metals - in elastic region allow/assume small relative deformations. Thus, for instance, for steel with the module/modulus of elasticity $E=2\cdot10^6$ of kg/cm² and tensile strength $\sigma_e=4\cdot10^3$ kg/cm² relative deformations are less than $\sigma_e/E=2\cdot10^{-3}$.

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This means that any elastic deformation of steel shell is accompanied by relative change in the metrics of its median surface less than $2\cdot10^{-3}$. Therefore, if this deformation leads to considerable changes in the form of shell, then it is almost geometric bending.

Further, the normal conditions of the attachment of the edge of shell guarantee geometric the nondeformability of its median surface

in the class of regular surfaces. Therefore the bending, which correspond to supercritical deformations, belong to a broader class of piecewise-regular surfaces. This means that the surface, which reproduces the form of shell during supercritical deformation, must have fin/edges. On the surface of shell, these fin/edges are smoothed.

The nearness of the supercritical deformation of shell to certain of its isometric conversion with special feature/peculiarities along lines (fin/edges) creates the specific specific character in the energy distribution of deformation according to the surface of shell. Namely, it noticeably is concentrated in the vicinity of fin/edges. The considerations, based on variation principle, make it possible to refine the form of the deformed shell near fin/edges and to determine strain energy in the vicinity of fin/edges depending on the geometric values, which relate to fin/edge. As a result functional U (strain energy) proves to be determined on isometric transformations of median surface, which reproduce the form of shell during supercritical deformation. So we come to the following variation principle A.

Considerable supercritical deformation elastic-and shells under the action of the given load is close to that form of the isometric conversion of initial surface, which communicates steady-state value

to the functional

$$W = U_{\tilde{F}} - A_{\tilde{F}}.$$

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This functional is determined during piecewise-regular isometric transformations $\tilde{F}$ of median surface of shell. Its term/component/addend $U_{\tilde{F}}$ (strain energy) is determined by the following formula:

$$U_{\tilde{F}} = \frac{E\delta^3}{24(1-\nu^2)} \int\limits_{\tilde{F}}\int (\Delta k_1^2 + \Delta k_2^2 + 2\nu\,\Delta k_1\,\Delta k_2)\,d\sigma + \\ + cE\delta^{5/2} \int\limits_{\tilde{\gamma}} \frac{\alpha^{5/2}}{\rho^{1/2}}\,ds_\gamma + \frac{E\delta^3}{12(1-\nu^2)} \int\limits_{\tilde{\gamma}} \alpha(-2k + k_e + k_i)\,ds_\gamma.$$

Here Δk_1 and Δk_2 - main changes normal of curvatures with transition from the initial form of shell F to the isometric transformation $\tilde{F}$; 2a - angle between the tangential planes of surface $\tilde{F}$ along fin/edge (fin/edges) $\tilde{\gamma}$; ρ - radius of curvature curved γ; k_e and k_i - normal surface curvatures $\tilde{F}$ in the direction, perpendicular to fin/edge $\tilde{\gamma}$, k - normal surface curvature F in the appropriate direction; δ - thickness of shell, E - modulus of elasticity, and ν - Poisson ratio. Constant c≃0.19. Integration in first term is fulfilled by surface area $\tilde{F}$, and in remaining two term/component/addends - according to arc curved $\tilde{\gamma}$.

Term/component/addend $A_{\tilde{F}}$ represents the produced by external

load work by the deformation of shell of form of F in $\tilde{F}$ and is calculated as usually.

Elastic deformations of shell in the form, close to $\tilde{F}$, determined by principle A, are accompanied by appearance on its surface of considerable voltage/stresses from curvature in the vicinity of fin/edges. Maximum voltage/stress σ, connected with this curvature, being determined by the formula

$$\sigma = c'E\frac{\delta^{1/2}\alpha^{1/2}}{\rho^{1/2}}, \quad c' \simeq 0.9.$$

Thus, the application/use of principle A makes it possible to find the form of shell in basic approach/approximation and the appearing on its surface maximum voltage/stresses. but this is a solution of basic task.

The application/use of principle A makes it possible to explain, as changes the received by shell load during supercritical deformation and, thus, to find the smallest received by shell load.

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It is called lower critical. The determination of this load is of considerable interest on the following reason. Real shells are geometrically inadequate, inadequate also the loading of shell. Both

these of factor descend, besides is very not defined, the upper critical value of load. In connection with this in order to completely exclude the possibility of the loss of stability of shell, it is necessary to be oriented to lower critical load. Lower critical load is connected with the considerable deformations of shell and therefore it is less sensitive to the imperfection of geometric form and the method of loading. The determination of the elastic supercritical states of shell makes it possible to give natural lower limit for the effect of the geometric imperfections of its form on the value of upper critical load.

It must be noted that the information of common/general/total variation principle in the case of supercritical deformations to principle A assumes the solution of the task of the isometric transformations of initial surface. This task is also sufficiently difficult. However, in a number of concrete/specific/actual cases, geometric and other considerations of qualitative character make it possible to considerably narrow the class of the isometric transformations during which it is necessary to examine functional W. Thus, for instance, studying a question concerning the supercritical deformations of the flat strictly convex hulls, attached on edge, it is the possible to be restricted to the simplest isometric transformations - mirror bulge. Mirror bulge consists of the mirror reflection of the arbitrary segment of surface in the plane of its

basis/base.

The investigation of the supercritical deformations of real ones, therefore, limitedly elastic shells is impossible without the account to the limited elasticity of the material of shell, since appearing during such deformations voltage/stresses, as a rule, are located beyond elastic limit of material. The supercritical deformations of such shells are characterized by some specific character. It proves to be that the deformation, connected with the displacement of fin/edge over surface, stops at the onset on this fin/edge of plastic deformations. In connection with this principle A for shells with the limited elasticity is supplemented by the following condition.

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Limitedly elastic shell allow/assumes only such supercritical deformations, determined by principle A, with which voltage/stress σ on fin/edges, computed on the formula

$$\sigma = c'E\frac{\delta^{1/2}\alpha^{1/2}}{\rho^{1/2}},$$

is not exceeded time/temporary strength of materials σ_{θ}. (We we consider that the tensile strength is elastic limit.). The investigation of supercritical states for limitedly elastic shells taking into account the supplement indicated to principle A leads to

the qualitatively new results which make it possible to explain the data of the corresponding experiments.

As noted above, there is considerable interest in the determination of the load, by which the shell loses stability. The proposed by us method makes it possible to match up the solution and this task. The fact is that the received by shell load at the moment of loss of stability is stationary, and upon transfer to supercritical deformations it does not virtually change, while the form of shell changes very substantially. In connection with this it is the possible to define upper critical load as the load, received by shell with considerable bulge. Investigation in this plan/layout as a result leads to which follows principle V.

If the effective on shell load critical, then variation problem for the functional

$$W = U - A$$

on disruptive infinitely small bending of median surface has nontrivial solution, that is the bending field, which is the solution, not is equally identical to zero. Functional W is determined in the fields infinitesimal bending with breaks along the lines where are satisfied the conditions of the coupling

$$\tau - \tau' = \varkappa e.$$

Here difference of τ-τ' is break of the bending field, and e - unit

vector of the binormal curved γ, along which occurs the breakage.

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Addend U of functional W (strain energy) is calculated from the formula

$$U = \int_{\gamma} \frac{2E\delta^2\alpha^2\varkappa}{\sqrt{12}(1-\nu^2)\rho} ds,$$

where ρ, α, δ, E and ν have previous value, $\varkappa$ - the value of breakage, and integration is executed according to arc of curve (curves) γ. Addend A of functional W is calculated by the usual method as the produced by external load work by the deformation, given by the bending field.

Now about contents of the book. It consists of three chapters and two supplements. In Chapter One, are studied the supercritical deformations of strictly convex hulls, that is shells with positive Gaussian curvature. Chapter begins with reminding of some geometric facts, which relate to the bending of convexs surface. Then follows the substantiation of principle A and its application/use to the investigation of the supercritical elastic states of shell. In particular, thoroughly is examined a question concerning the supercritical deformations of the flat, rigidly attached concerning edge shells with the different methods of loading. Basic results are

compared with the data of the thoroughly placed experiments. Especially are examined the supercritical deformations limitedly elastic shells. At the end of the chapter, is investigated questions of the stability of the axially symmetric deformations of spherical shell with axially symmetric loading. As a result of this investigation, are explained the specific conditions for the applicability of the obtained in Chapter 1 results.

The second chapter is dedicated to the investigation of the loss of stability of strictly convex hulls. It begins with the substantiation of principle V and with its application/use to the task of the stability of flat, attached with respect to edge, strictly convex hull at a uniform external pressure (§1). In § 2 is contained the geometric study of the special infinitesimal bending of strictly convex surface, while in § 3 results of this investigation, are applied during the determination of critical loads for the shells of revolution with the different methods of loading (external pressure, internal pressure, twisting).

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In the third chapter are examined the supercritical deformations of cylindrical shells in basic load cases (axial compression, external pressure, twisting). Investigations are based on the

application/use of principle A. Here are determined lower critical loads for unlimitedly elastic shells and for shells with the limited elasticity, and also is studied the effect of the initial bending of shell on stability. The results of theoretical studies also are compared with the data of experiments.

In supplement I, are examined some questions of the dynamics of shells. The methods of the study of the static tasks of the theory of shells, developed in the book, in a number of cases can be successfully used to the tasks of dynamics. In supplement this is illustrated based on specific examples.

In supplement II, are studied the isometric transformations of cylindrical surfaces, which correspond to supercritical deformations with ridging to entire length of shell; are given the geometric and analytical descriptions of these transformations.

The concrete/specific/actual results, which are contained in this book, in essence are published in the separate issues of the publishing house of Kharkov university [1-7]. In the present presentation of an inaccuracy in the preceding/previous publications, they are removed, and the main thing are formulated the common/general/total principles, with the aid of which are obtained concrete/specific/actual results (principles A and B).

The reading of the proposed book does not require large knowledge in the theory of shells; however, is assumed known geometric culture. Basic results are formulated, as a rule, it is completely available. They are represented either by the appropriate formulas or graphs. In this sense the book is available to the wide circle of the readers.

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Chapter One.

STRICTLY CONVEX SHELLS DURING SUPERCRITICAL DEFORMATIONS.

The proposed investigation of elastic shells during supercritical deformations we will begin from the examination of strictly convex hulls. This of shells whose median surface has positive normal curvature according to any direction. First of all, we will base the certain common/general/total principle A, which by standard form is applied during the solution of different tasks. This principle reduces the solution of the task of the elastic supercritical states of shell to the examination of variational problem for the functional, determined during piecewise-regular isometric transformations of the initial form of shell. Then we will pass to the solution of specific problems and compare the obtained results with the data of the corresponding experiments.

Real shells, possessing the limited elasticity, in supercritical state, as a rule, experience/test elastoplastic deformations. In connection with this principle A is more precisely formulated in connection with the case of the shells, which possess the limited

elasticity.

The proposed method of the study of supercritical deformations allow/assumes vast, but all the same limited field of application to real shells. We have this in form and therefore let us attempt to outline this region by the appropriate conditions.

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§ 1. Energy of elastic deformation of shell in supercritical stage.

Elastic deformations of shell are accompanied by small changes in the metrics of its median surface. Therefore, if this deformation of shell leads to considerable changes in its exterior form, then it (form) is determined in essence by the geometry of its initial state and close to isometric transformation. In connection with this we let us recall some geometric facts, which relate to the bending of surfaces.

1. Supercritical deformation and geometric bending. F - regular (at least twice differentiated) surface. This means that on surface can be introduced the curvilinear coordinate grid/network u, v so that vector function r(u, v), the assigning surface in these coordinates, is regular (at least twice differentiated) function. The

linear cell/element of surface, which corresponds to this parametrization u, v, is called the differential quadratic form

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

where

$$E = r_u^2,\quad F = r_u r_v,\quad G = r_v^2.$$

The surfaces, in which with corresponding parametrizations u, v linear cell/elements are identical, are called isometric. Geometrically this means that there is conformity of points of these surfaces, by which any two corresponding curves on these surfaces have identical lengths. The geometric property indicated can be accepted as the determination of the concept of isometry. In such a mind it keeps sense, also, for irregular surfaces.

If among the surfaces of this class each surface, isometric F, are equal to F, then surface F is called uniquely determined in this class. For example, any closed convex surface (even without assumption about regularity) is unambiguously determined in the class of convexs surface [8].

The indication of the class of the surfaces in question is substantial. One and the same surface can be unambiguously determined in one class of surfaces and not be at the same time unambiguously determined in other, broader class. So, the closed convex surface is not unambiguously determined in the class of piecewise-convex

surfaces.

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Really/actually, F - closed convex surface. Let us conduct the plane α, which intersects surface of F, and it is reflected one of its parts into which it is divide/marked off by plane α, it is mirror in this plane (Fig. 1). The closed surface F*, comprised of part S_2 of initial surface and S_1^* - mirror reflection S_1 in plane α - is isometric surface F. Isometric conformity consists of comparison to each point P of surface F, of belonging S_2, coinciding with it point of surface F*, and to point P, which belongs S_1 - point P*, being mirror image P in plane α. It is obvious, this representation F on F* is isometric. But surfaces F and F* are not knowingly equal, since there is this motion or no motions and mirror reflection for the entire surface of F (but not its separate parts), which would combine it with surface of F*. Let us agree to call the examined isometric transformation of surface of F into F* mirror bulge.

If we F - this surface and F' - surface, isometric F, then they speak also, that F' is obtained by the geometric bending (or it is simple by bending) from F. Sometimes under bending is understood the continuous deformation of surface F in F' with the preservation/retention/maintaining of isometry at each moment of

deformation. We will use word "bending" both in that and in other sense, more precisely formulating it when this can lead to misunderstandings. Let us note that in the examined example of mirror the bulge of convex surface F surface F* can be obtained by continuous bending from F. For this it is sufficient to take plane α, first not intersecting surface, and then to move it for surface, fulfilling in each position the construction indicated with the mirror reflection of the intercept/detached part.

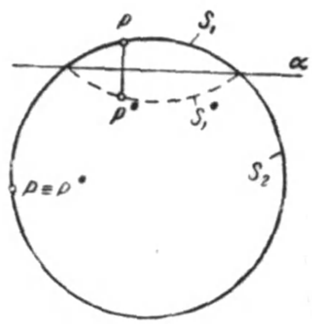

Fig. 1.

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In connection with the forthcoming application/appendices for us special interest they represent the bending of strictly convex regular surfaces with edge under the condition of the immobility both of the points of edge and tangent planes of surface at these points. For such surfaces we will, first of all, establish/install their unique determination in the class of the twice differentiated surfaces.

F - twice differentiated strictly convex surface with edge γ. It is not difficult to supplement it to certain closed convex surface Φ, for example, after taking the convex hull of surface F. If we surface F with attached edge γ indicated it allow/assumed nontrivial

isometric transformation in class regular of surfaces, then the closed surface Φ, obviously, would allow/assume isometric transformation in the class of convex surfaces. But this is impossible in view of the theorem about the unique determination for such surfaces. Affirmation is proved.

As noted above, surface F, being rigid/inflexible in one class of surfaces, can be bent in a broader class. In particular, regular, attached along the edge strictly convex surface is nonbendable in the class of regular surfaces, but bendable in the class of piecewise-regular surfaces. Of this, us convinces an example of mirror bulge. Here isometric transformation is connected with the disturbance/breakdown of regularity (by formation of fin/edge) along certain curved, that limits convex region on surface. In connection with this let us examine the following question. The how most common/general/total isometric transformation of regular, attached on edge, over strictly convex surface in the class of piecewise-regular surfaces, if the disturbance/breakdown of regularity you do solve along assigned/prescribed curve γ, which limits the convex region G on surface of F?

G* - part of surface F, arranged/located out of region G and adjoining the edge. First of all, we confirm that during any isometric transformation of surface of F with the

disturbance/breakdown of the regularity only of lengthwise curve γ surface G' they will not change, that is its points remain fixed. This escape/ensues from the uniqueness of the solution of the Cauchy problem for the differential equation to examination of which is reduced the task of the construction of surface, isometric to given.

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This is the equation of Monge-Ampere of elliptical type. The attachment of edge of surface gives initial conditions for the Cauchy problem indicated. The inalterability of region G' will entail the inalterability of its edge γ. Thus, during the isometric transformation of surface of F is deformed only part G, moreover γ - an edge of region G - remains fixed. Let during the isometric transformation F into $\bar{F}$ its part G transfer/convert in $\bar{G}$.

If surface $\bar{G}$ is directed by convexity to the same side, as G, then $\bar{F}$ will be convex surface. It is not difficult to conclude that in this case it must coincide with F. For this, it suffices to use the reasoning, with the aid of which is establish/installed unique determination F in the class of regular surfaces. Thus, if surface F allow/assumes nontrivial isometric transformation, then it is necessary to count that $\bar{G}$ is directed by convexity in the other direction. In this case, surfaces G and $\bar{G}$, having overall edge γ,

compose the closed convex surface (Fig. 2). Let us designate it Φ. Surface Φ allow/assumes isometric mapping onto itself. This representation consists of comparison to each point P of region G of the corresponding according to isometry point of region $\bar{G}$ and each point P of region $\bar{G}$ of the corresponding on isometry point of region G.

In view of the unique determination of the closed convexs surface, the constructed isometric representation of surface Φ onto itself must be reduced to motion or to motion and mirror reflection. Since points curved γ during isometric representation remain fixed, the matter is reduced to the mirror reflection of surface Φ relative to a certain plane. Curve γ, being fixed, must lie/rest at this plane. Thus, we come to following conclusion.

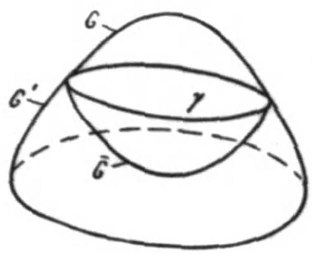

Fig. 2.

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Isometric transformation of strictly convex regular surface, attached on edge, in the class of piecewise-regular surfaces with disturbance of regularity only by lengthwise curve γ, which limits the convex region G, is possible only if curved γ flat/plane, and in this case it is reduced to the mirror reflection of region G in plane by curve γ.

Let us turn now to the supercritical deformations of elastic shells. First of all, let us explain the concept of supercritical deformation. By supercritical deformation we will understand deformation, with which the shell experience/tests considerable changes in exterior form. Such deformations appear usually as a result of the loss of stability of the shell when the effective load

reaches critical value. Hence and name - supercritical deformations.

The materials from which are manufactured the shells, as a rule, do not allow/assume considerable internal strains. For example, for steel with tensile strength $\sigma_{e}=4\cdot10^{3}$ kg/cm² and module/modulus of elasticity $E=2\cdot10^{6}$ of kg/cm² relative elongation (compression) ε in the region of elastic deformations does not exceed value

$$\frac{\sigma_{e}}{E}=0{,}002.$$

Approximately the same relationship/ratio occurs for other most widely used structural/design materials. Hence we draw the following important conclusion.

D → ~~d~~uring the supercritical deformation of elastic shell its median surface experience/tests in essence isometric transformation.

One should, however, note that this conclusion has real value only if the discussion deals with considerable ones change in the form of shell during deformation. But if the deformed shell has form, close to initial, then our conclusion/derivation is trivial and is the consequence of this nearness.

Let strictly convex shell with regular surface, attached on edge, under the action of certain load be deformed with bulge on the convex region G. let us explain the form of the deformed shell,

assuming bulge considerable.

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Since the supercritical deformation of shell exists in essence geometric bending, and any bending of the attached on edge convex surface with the bulge of convex region is reduced to mirror bulge, then we consist:

The considerable supercritical deformation of the attached in edge strictly convex hull is unavoidably close to the appropriate form of mirror bulge.

In view of the approach/approximation of supercritical elastic deformation indicated by mirror bulge, the requirement so that the change in the form of shell would be considerable, is actually reduced to that so that the region of bulge would encompass the significant part of the shell. However, as we will see more lately, for real shells due to the limited elasticity of material, the region of bulge proves to be in a specific manner limited. As a result our conclusion about the approach/approximation of elastic bulge to mirror has real value only for the shells of small size/dimensions or, with these size/dimensions, for sufficiently slightly curved shells. In connection with this our examinations in present chapter

will be related to flat strictly convex hulls.

2. Energy of elastic deformation of shell. Formulation of principle A. The elastic shell F with the regular surface under the action of certain load, which more precisely formulate we will not, is experience/tested supercritical deformation, taking form of $\bar{F}$. If median surface of the shell is geometrically nonbendable in the class of regular surfaces, then the deformed shell $\bar{F}$ is close to corresponding to form $\tilde{F}$ of isometric transformation F with the disturbance/breakdown of regularity along some lines $\tilde{\gamma}$ and the formation of the fin/edges along these lines. The presence of special feature/peculiarities in the form of fin/edges during the isometric transformation $\tilde{F}$ of surface of F and the nearness of surface $\tilde{F}$ to $\bar{F}$ give grounds to speak about fin/edges (smoothed fin/edges) on the deformed surface of shell $\bar{F}$. It goes without saying, their form and position are determined only in the known approach/approximation, which depends on the nearness of the deformed shell $\bar{F}$ to surface of $\tilde{F}$. So that to conditional fin/edges $\bar{\gamma}$ on the surface of the deformed shell to ascribe the specific form and position, we will act as follows.

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To fin/edge $\tilde{\gamma}$ on surface of $\tilde{F}$ on isometry it corresponds certain

curved γ on the initial surface of F. During the deformation of this curve in question on the deformed shell, corresponds curve $\bar{\gamma}$. This curve logical to accept for conditional fin/edge.

Our most immediate task consists of the determination of energy of elastic deformation upon transfer from one F to ~~next~~ $\bar{F}$. This energy U is represented by advisable to break into two parts - U' and U". By U' we will understand energy of deformation over the basic surface of shell out of the vicinity of fin/edges $\bar{\gamma}$, while by U" - energy of deformation within the vicinities indicated.

Relative to energy U' we will assume that it consists in essence of energy of bending, and therefore per the unit surface area it is determined from known to the formula

$$\overline{U}' = \frac{D}{2}\left(\Delta k_1^2 + \Delta k_2^2 + 2\nu\,\Delta k_1\,\Delta k_2\right),$$

where D - flexural rigidity of shell, and Δk_1 and Δk_2 - main changes in the normal curvatures of shell during its deformation into form of $\bar{F}$. In view of the fact that the disturbance of the regularity of surface $\tilde{F}$ occurs only on fin/edges $\tilde{\gamma}$, it is possible to count that surfaces $\tilde{F}$ and $\bar{F}$ out of the vicinity of fin/edges not it is only point close, but have also close normal curvatures according to the corresponding directions. Hence it follows that in formula for $\overline{U}'$ value Δk_1 and Δk_2 it is possible to consider changes in the normal curvatures upon transfer from surface of F to the isometric

transformation $\tilde{F}$.

In order to obtain energy U', it is necessary to integrate expression $\bar{U}'$ with respect to surface area $\tilde{F}$, eliminating the vicinities of fin/edges. In this case, if vicinities are small, as this we will assume, then integration can be extended to entire surface of $\tilde{F}$.

Therefore

$$U' = \frac{D}{2} \int_{\tilde{F}} \int (\Delta k_1^2 + \Delta k_2^2 + 2\nu \Delta k_1 \Delta k_2) \, d\sigma.$$

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Let us turn now to strain energy U" in the vicinity of fin/edges. P - arbitrary point of fin/edge $\tilde{\gamma}$ on surface of $\tilde{F}$. Being limited to examination near this point, let us introduce cylindrical coordinate system ϕ, r, z, after accepting as the axle/axis of system the straight line, passing through the center of the touching circle curved $\tilde{\gamma}$ at point P of perpendicular to the plane this circle. Let us isolate two radial planes, by the close P, the cell/element of shell of vicinity by curve $\tilde{\gamma}$ is computed in it strain energy (Fig. 3). From demonstrative considerations about the deformation of shell in the vicinity of fin/edge, we consist that the strain energy of the chosen

cell/element it is consisted in essence from energy of curvature in the plane, perpendicular to fin/edge and energy of expansion-compression in the direction of fin/edge.

Let the section/cut of surface of $\tilde{F}$ by the plane, perpendicular to fin/edge at point P, in coordinates r, z be assigned by the equation

$$z = z(r).$$

Let us designate through u and v shifts of the points of surface $\tilde{F}$ during its deformation in $\bar{F}$: u - on the principal normal, and v - on binormal curved $\tilde{\gamma}$ at point P. Then, if the tangential planes of surface $\tilde{F}$ by lengthwise curve $\tilde{\gamma}$ form small angle, then change in the normal curvature upon transfer from surface of $\tilde{F}$ and $\bar{F}$ in the direction, perpendicular to fin/edge, it will be it is equal

$$\Delta_1 k \simeq v'',$$

where the differentiation is conducted according to to the variable r.

Fig. 3.

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If now a change of the normal curvature in the same direction upon transfer from the initial form of F to $\tilde{F}$ is designated Δk, then the total change in the normal curvature during deformation F in $\bar{F}$ will be equal

$$\Delta_2 k = v'' + \Delta k.$$

Energy of curvature U" , the chosen cell/element is determined from the formula

$$U_1'' = \frac{D}{2} \int \int (v'' + \Delta k)^2 \, d\sigma,$$

where the integration is fulfilled by the area of cell/element.

Relative tensile strain - the compression of median surface of shell in the direction, perpendicular to the section/cut in question, obviously, it will be equal to

$$\varepsilon \simeq \frac{u}{\rho},$$

where ρ - a radius of curvature of fin/edge $\breve{\gamma}$ at point P. Hence for strain energy U''_2, connected with elongation (compression) median surface of the chosen cell/element, is obtained the expression

$$U_2'' = \frac{D'}{2} \int \int \frac{u^2}{\rho^2} d\sigma,$$

where D' - rigidity of shell to elongation (compression).

Let us designate through $2\bar{\varepsilon}$ the width of the vicinity of fin/edge $\tilde{\gamma}$ in question. Then total energy of the deformation of the chosen cell/element of shell can be represented in the form

$$U'' = \left\{ \frac{D}{2} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} (v'' + \Delta k)^2 ds + \frac{D'}{2} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \frac{u^2}{\rho^2} ds \right\} \Delta l,$$

where Δl - width of cell/element in direction $\breve{\gamma}$. $\bar{U}''$ - strain energy in the vicinity of fin/edge per the unit of its length. Then

$$\bar{U}'' = \frac{D}{2} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} (v'' + \Delta k)^2 ds + \frac{D'}{2} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \frac{u^2}{\rho^2} ds.$$

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Here Δk - change of the normal curvature in the direction, perpendicular to fin/edge, upon transfer from the initial form of F

to $\tilde{F}$, and integration is fulfilled in width 2ε of the vicinity of fin/edge $\tilde{\gamma}$.

Since we disregard the deformation of median surface in the direction, perpendicular to fin/edge, the variables u, the v, which are determining deformation shells, must satisfy certain condition. This condition we will obtain, equalizing the linear cell/elements $d\tilde{s}^2$ and $d\bar{s}^2$ surfaces $\tilde{F}$ and $\bar{F}$ in the section/cut, perpendicular to fin/edge. We have

$$d\tilde{s}^2 = dr^2 + dz^2,$$
$$d\bar{s}^2 = (dr + du)^2 + (dz + dv)^2.$$

Hence

$$dr\,du + dz\,dv + \frac{1}{2}(du^2 + dv^2) = 0.$$

Let us designate through 2α the angle between the tangential planes of surface $\tilde{F}$ along fin/edge $\tilde{\gamma}$. Since the geodetic curvatures of fin/edge $\tilde{\gamma}$ in surface of $\tilde{F}$ differ only in terms of sign with approach to $\tilde{\gamma}$ from two sides, then the osculating plane of fin/edge forms to by the tangential planes of the surface of identical ones the angles, equal to α. Under the assumption of the smallness of angle α the relationship/ratio between displacements u, v can be simplified. Specifically,, noting that

$$\left|\frac{dz}{dr}\right| \simeq \alpha \quad и \quad u'^2 \ll |u'|,$$

we can write this relationship/ratio in the following form:

$$u' \pm \alpha v' + \frac{1}{2} v'^2 = 0.$$

Here sign "+" must be taken in one half-neighborhood, and into another - sign "-".

Let us speak, that surfaces F and $\tilde{F}$ at the appropriate by isometry points P and $\tilde{P}$ are equally oriented, if the circuit/bypasses of these points, which correspond on isometry, form with the direction of external standard simultaneously right (or left) screw/propeller.

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But if for one surface there will be right screw/propeller, and for another left, then we will speak, that at such points of surface they are oriented oppositely. Since upon transfer through the fin/edge of surface $\tilde{F}$ the direction of convexity changes, then in one half-neighborhood of the fin/edge of surface F and $\tilde{F}$ they are oriented equally, but into another - it is opposite. Let us agree to call external that half-neighborhood in which surfaces are oriented identically, and internal - that in which they are oriented oppositely.

Let us select now the direction of Z-axis of cylindrical coordinate system so that the relationship/ratio between displacements u, V in external half-neighborhood would be

$$u' + av' + \frac{v'^2}{2} = 0.$$

In this case, naturally, in internal half-neighborhood it will be

$$u' - \alpha v' + \frac{v'^2}{2} = 0.$$

Now k - normal curvature of initial surface in the direction, perpendicular to fin/edge (k>0), k_e - normal surface curvature $\widetilde{F}$ in the appropriate direction from the side of the external half-neighborhood of fin/edge $\tilde{\gamma}\,(k_e > 0)$ and k_i - normal surface curvature $\widetilde{F}$ also in the direction, perpendicular $\widetilde{\gamma}$, but from the side of internal half-neighborhood $(k_i < 0)$. Then value Δk, which enters into formula for $\bar{\bar{U}}''$, in external half-neighborhood is equal to $k - k_e$, and $k - k_i$ - in internal.

Expression for strain energy $\overline{U}''$ is expedient to convert as follows:

$$\overline{U}'' = \frac{D}{2}\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} v''^2\,ds + \frac{D'}{2}\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \frac{u^2}{\rho^2}\,ds + D\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \Delta k v''\,ds + D\bar{\varepsilon}\,(\Delta k)^2.$$

Last/latter term/component/addend in this expression we will reject/throw immediately, assuming small width $2\bar{\varepsilon}$ of the vicinity of fin/edge, in which are conducted our examinations. Let us turn to third term/component/addend.

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Since function V' suffers breakage on fin/edge (that is with s=0), then

$$\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \Delta k v''\, ds = -\Delta k_e v'(+0) + \Delta k_i v'(-0) + \\ + \Delta k_e v'(\bar{\varepsilon}) - \Delta k_i v'(-\bar{\varepsilon}).$$

Last/latter two add/composed they can be lowered, since V'(s) unlimitedly decreases during distance/removal from fin/edge. As far as terms are concerned first two, when little α them it is possible to consider equal - $\alpha\Delta k_e$ and $\alpha\Delta k_i$. Taking all this into consideration, we can write expression for strain energy in the following form:

$$\bar{U}'' = \frac{D}{2}\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} v''^2\, ds + \frac{D'}{2}\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \frac{u^2}{\rho^2}\, ds - D\alpha(\Delta k_e + \Delta k_i).$$

If we here introduce values Δk_e and Δk_i, then we will obtain

$$\bar{U}'' = \frac{D}{2}\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} v''^2\, ds + \frac{D'}{2}\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \frac{u^2}{\rho^2}\, ds + D\alpha(-2k + k_e + k_i).$$

The found by us expression for strain energy $\bar{U}''$ in the vicinity of fin/edge depends on two functions u and V, connected by the relationship/ratio

$$u' \pm \alpha v' + \frac{v'^2}{2} = 0.$$

We will determine these functions, and with them and strain

energy, minimizing functional $\bar{U}''$ $(u, \check{v})$ [1].

FOOTNOTE [1]. This escape/ensues from the common/general/total variation principle to which is reduced the solution of the task of the elastic equilibrium of shell. The corresponding task by us is dismembered to two parts: the determination of the form of shell near fin/edges and the subsequent determination of the position of fin/edges themselves. ENDFOOTNOTE.

Let us introduce instead of the variables u, V, s new the variables $\bar{u}$, $\bar{v}$ and $\bar{s}$, set/assuming

$$\bar{u} = \frac{u}{\varepsilon\rho\alpha^2}, \quad \bar{v} = \frac{v'}{\alpha}, \quad \bar{s} = \frac{s}{\rho\varepsilon},$$

where

$$\varepsilon^4 = \frac{\delta^2}{12\rho^2\alpha^2}.$$

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Noting that

$$D = \frac{E\delta^3}{12(1-\nu^2)}, \quad D' = \frac{E\delta}{1-\nu^2},$$

for strain energy $\bar{U}''$, in new variables we will obtain the following expression:

$$\bar{U}'' = \frac{E\delta^{5/2}\alpha^{5/2}}{12^{3/4}(1-\nu^2)\rho^{1/2}} J + D\alpha(-2k + k_e + k_l),$$

where

$$J = \frac{1}{2}\int_{-\bar{s}^*}^{\bar{s}^*} (v'^2 + u^2)\, ds.$$

and

$$\bar{\varepsilon}^{*}=\frac{\bar{\varepsilon}}{\rho\varepsilon}.$$

The feature above the designations of new variables for simplicity of recording is lowered.

The relationship/ratio between displacements in new variables takes the form

$$u'\pm v+\frac{1}{2}v^{2}=0.$$

The limits of integration $\pm\bar{\varepsilon}^{*}$ in expression for J depend on parameter ε, when $\varepsilon\to 0$, $\bar{\varepsilon}^{*}\to\infty$. In connection with this, being limited to the case of such shells and their deformations, for which this parameter is low, let us replace integration limits in J on +-∞:

$$J=\frac{1}{2}\int_{-\infty}^{\infty}(v'^{2}+u^{2})\,ds.$$

Now task regarding u, V, and energies of deformation $\bar{U}^{*}$ is reduced to task to the minimum for functional J in the nonholonomic constraint

$$u'\pm v+\frac{v^{2}}{2}=0.$$

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So that our variational problem would become completely determined, we must still place boundary conditions for the varied functions u and v. These conditions logically escape/ensue from the fact that surface $\bar{F}$ far from fin/edge is sufficiently close to $\tilde{F}$. Therefore we set/assume

$$u(-\infty)=u(\infty)=0, \quad v(-\infty)=v(\infty)=0.$$

J_0 - minimum of functional J. Then strain energy $\bar{U}''$ is determined from the formula

$$\bar{U}''=\frac{E\delta^{5/2}a^{5/2}J_0}{12^{5/4}(1-\nu^2)\rho^{1/2}}+Da(-2k+k_e+k_i).$$

Strain energy in all vicinities of fin/edge $\tilde{\gamma}$ is obtained by the integration of this expression for arc curved $\tilde{\gamma}$. Let us note that the obtained expression of strain energy depends only on the geometric characteristics of surface $\tilde{F}$ on fin/edge $\tilde{\gamma}$.

The definition of the states of the elastic equilibrium of shell, which is located under the action of the given load, as is known, it is reduced to the solution of task to the extremum of the functional

$$W=U-A,$$

where U energy of elastic deformation of shell, A - the produced by external load work by this strain. Shell under the action of the given load takes such form of $\bar{F}$ which communicates to functional W steady-state value.

For energy of elastic supercritical strain, we found the expression

$$U = U'(\tilde{F}) + U''(\tilde{F}).$$

depending only on surface of $\tilde{F}$, isometric F and close to $\bar{F}$. In view of the nearness of surface $\tilde{F}$ to $\bar{F}$, it is possible to count that the produced by external load work A is determined by strain F in $\tilde{F}$, i.e., that

$$A(\bar{F}) \simeq A(\tilde{F}).$$

Now we can formulate the following principle A.

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The considerable supercritical deformation of elastic shell under the action of the given load is close to that form of the isometric conversion of initial surface, which communicates steady-state value to the functional

$$W = U(\tilde{F}) - A(\tilde{F}).$$

This functional is determined during the isometric transformations of median surface of shell. Addend $U(\tilde{F})$ is determined by the following

formula:

$$U(\tilde{F}) = \frac{E\delta^3}{24(1-\nu^2)} \iint\limits_{\tilde{F}} (\Delta k_1^2 + \Delta k_2^2 + 2\nu \Delta k_1 \Delta k_2)\, d\sigma +$$

$$+ cE\delta^{5/2} \int\limits_{\tilde{\gamma}} \frac{\alpha^{5/2}}{\rho^{1/2}}\, ds_\gamma + \frac{E\delta^3}{12(1-\nu^2)} \int\limits_{\tilde{\gamma}} \alpha(-2k + k_e + k_i)\, ds_\gamma.$$

Here Δk_1 and Δk_2 - main changes in the normal curvatures upon transfer from the initial form of shell F and to the isometric transformation $\tilde{F}$; 2α - angle between the tangential planes of surface $\tilde{F}$ along fin/edge (fin/edges) $\tilde{\gamma}$; ρ - radius of curvature curved $\tilde{\gamma}$; k_e and k_i - normal surface curvatures $\tilde{F}$ in the direction, perpendicular to fin/edge $\tilde{\gamma}$, k - normal surface curvature F in the appropriate direction; δ - thickness of shell, E - modulus of elasticity, ν - Poisson ratio. Constant

$$c = \frac{J_0}{12^{5/4}(1-\nu^2)}.$$

Integration in first term is fulfilled by surface area $\tilde{F}$, and in remaining two term/component/addends - according to the friend of fin/edges $\tilde{\gamma}$.

Addend $A(\tilde{F})$ is the produced by external load work by the deformation of shell into form of $\tilde{F}$ and is calculated by the usual method.

The proposed principle determines not only the form of shell during supercritical deformation, but also the maximum voltage/stresses in its material during this deformation.

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Really/actually, maximum voltage/stresses appear, obviously, in the vicinity of fin/edge they are caused either by curvature in the plane, perpendicular to fin/edge or by the elongation (compression) of median surface in perpendicular direction. In initial the variables u, V for a maximum voltage/stress σ' from curvature in the plane, perpendicular to fin/edge, we have

$$\sigma' \simeq \frac{E\delta}{2} \max|v''|.$$

The maximum tensile stress (compression) of median surface in the direction of fin/edge will be

$$\sigma'' \simeq E \max\left|\frac{u}{\rho}\right|.$$

Transfer/converting in these formulas to to the dimensionless variables $\bar{u}$, $\bar{V}$, we will obtain

$$\sigma' = c'E\frac{\delta^{1/2}\alpha^{3/2}}{\rho^{1/2}},$$

$$\sigma'' = c''E\frac{\delta^{1/2}\alpha^{3/2}}{\rho^{1/2}},$$

where c' and c'' - constants, determined with the aid of functions u, V, which realize the minimum of functional J, on the formulas

$$c' = \frac{12^{1/4}}{2} \max|v'|,$$

$$c'' = \frac{\max|u|}{12^{1/4}}.$$

The values of these constants, and also the constant c in functional W we will find in the following point/item.

3. Solution of variational problem for functional J. Determination of the constants c, c' and c''. Examining a question concerning energy of the elastic supercritical deformation of shell, we arrived at task to the minimum of the functional

$$J=\frac{1}{2}\int_{-\infty}^{\infty}(v'^2+u^2)\,ds$$

in the nonholonomic constraint

$$u'\pm v+\frac{v^2}{2}=0$$

and under following boundary conditions for the varied functions:

$$u(-\infty)=u(\infty)=0,\quad v(-\infty)=v(\infty)=0.$$

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By the sense of task its solution must be searched for among functions V, which have gap with s=0 with the condition

$$v(+0)-v(-0)=-2.$$

This condition escape/ensues from the corresponding condition for initial the variable V, where it takes the form

$$v'(+0)-v'(-0)=-2a.$$

It is natural to search for the solution of our variational problem in the class of the odd functions u and V. Then

$$J=\int_{0}^{\infty}(v'^2+u^2)\,ds.$$

Communication/connection takes the form

$$u' + v + \frac{v^2}{2} = 0.$$

Boundary conditions will be

$$u(0) = 0, \quad u(\infty) = 0, \quad v(0) = -1, \quad v(\infty) = 0.$$

Since V(s) $\to$ 0 with $s \to \infty$, then with large s communication/connection between u and V can be simplified, after reject/throwing unessential term/component/addend $v^2/2$. Then we obtain

$$u' + v = 0.$$

If we exclude function V, after expressing it through u taking into account communication/connection

$$u' + v + \frac{v^2}{2} = 0,$$

then functional J will take the form

$$J = \int_0^\infty f(u, u', u'')\, ds.$$

With large s

$$f \simeq u''^2 + u^2.$$

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Therefore the equation of Euler for our variational problem with large s takes the form

$$u^{IV} + u = 0.$$

Its general solution

$$u=\sum_k c_k e^{\omega_k s},$$

where ω_k - roots of the characteristic equation

$$\omega^4+1=0.$$

In view of condition $u(\infty)=0$, in expression for u must be only term/component/addends, which correspond to roots ω_k with negative real part, i.e., to the roots

$$\omega_1=-\frac{1}{\sqrt{2}}(1+i), \quad \omega_2=-\frac{1}{\sqrt{2}}(1-i).$$

Thus, with large s

$$u=c_1e^{\omega_1 s}+c_2e^{\omega_2 s},$$
$$v=-u'=-c_1\omega_1e^{\omega_1 s}-c_2\omega_2e^{\omega_2 s}.$$

Taking into account the obtained asymptotic representation (with large s) for functions u, V, let us search for V on an entire semi-axis in the form

$$v=\alpha_1x+\alpha_2y+\alpha_{11}x^2+2\alpha_{12}xy+\alpha_{22}y^2+\ \ldots,$$

where for brevity it is marked

$$x=e^{\omega_1 s}, \quad y=e^{\omega_2 s}.$$

Let us compose the equation of Euler for function V. According to eylera - Lagrange's method, the solution of task to the minimum for functional J in the assigned/prescribed nonholonomic constraint is equivalent to the solution of task to unconditional extremum for the functional

$$J'=\int_0^\infty\left(u^2+v'^2+\lambda(s)\left(u'+v+\frac{v^2}{2}\right)\right)ds.$$

The equations of Euler for this functional will be

$$\lambda(1+v)-2v''=0, \quad 2u-\lambda'=0.$$

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To these two equations it is necessary to connect the equation of relation

$$u'+v+\frac{v^2}{2}=0.$$

Integrating last/latter equation in limits (0, ∞) and noting that u(∞) =0, we will obtain

$$u=\int_s^{\infty}\left(v+\frac{v^2}{2}\right)ds.$$

From the equation

$$\lambda(1+v)-2v''=0$$

it follows that λ(∞)=0, since V(∞)=0 and, therefore, it is possible to count v''(∞)=0. Substituting the obtained integral representation for u in the equation

$$2u-\lambda'=0,$$

and integrating it in limits (s, ∞), we will obtain

$$\lambda=-2\int_s^{\infty}\int_t^{\infty}\left(v+\frac{v^2}{2}\right)dt\,ds.$$

But now with the aid of the equation

$$\lambda(1+v)-2v''=0$$

we obtain the unknown integrodifferential equation for V

$$v'' + (1+v)\int_s^\infty \int_t^\infty \left(v + \frac{v^2}{2}\right) dt\, ds = 0.$$

Substituting in this equation expression for V in the form a row/series according to degrees X, y, let us have

$$x\left(\alpha_1\omega_1^2 + \frac{\alpha_1}{\omega_1^2}\right) + y\left(\alpha_2\omega_2^2 + \frac{\alpha_2}{\omega_2^2}\right) + \\ + x^2\left(4\alpha_{11}\omega_1^2 + \frac{1}{4\omega_1^2}\left(\alpha_{11} + \frac{\alpha_1^2}{2}\right) + \frac{\alpha_1^2}{\omega_1^2}\right) + \ldots = 0.$$

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Equalizing to zero coefficients of X, y, x^2, ..., we will obtain infinite system of equations for α_1, α_2, α_{11}, α_{12}, ... The first two equations of this system are satisfied identically, since ω_1 and ω_2 are the roots of the equation

$$\omega^4 + 1 = 0.$$

Remaining equations allow to determine α_{11}, α_{12}, ... depending on α_1 and α_2.

After expressing thus the coefficients of expansion through α_1 and α_2, we will use for determining the latter the boundary conditions: u(0)=0, V(0)=-1. We have

$$u + \frac{\alpha_1}{\omega_1}x + \frac{\alpha_2}{\omega_2}y + \frac{x^2}{2\omega_1}\left(\alpha_{11}(\alpha_1, \alpha_2) + \frac{\alpha_1^2}{2}\right) + \ldots = 0.$$

Hence, set/assuming s=0, we will obtain

$$\frac{\alpha_1}{\omega_1}+\frac{\alpha_2}{\omega_2}+\frac{1}{2\omega_1}\left(\alpha_{11}(\alpha_1,\ \alpha_2)+\frac{\alpha_1^2}{2}\right)+\ \dots=0.$$

The second equation for α_1, α_2 is obtained from boundary condition V(0)=-1:

$$\alpha_1+\alpha_2+\alpha_{11}(\alpha_1,\ \alpha_2)+\ \dots\ =-1.$$

The obtained system of equations for α_1, α_2 is conveniently presented in the form

$$\frac{\alpha_1}{\omega_1}+\frac{\alpha_2}{\omega_2}+P(\alpha_1,\ \alpha_2)=0,$$
$$\alpha_1+\alpha_2+Q(\alpha_1,\ \alpha_2)=-1.$$

For solving this system, it is possible to use the method of successive approximations. The first approximation is obtained by the solution of the system

$$\frac{\alpha_1^{(1)}}{\omega_1}+\frac{\alpha_2^{(1)}}{\omega_2}=0,\quad \alpha_1^{(1)}+\alpha_2^{(1)}=-1.$$

For obtaining the second approach/approximation, let us substitute obtained values of $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ in P and Q we solve the system

$$\frac{\alpha_1^{(2)}}{\omega_1}+\frac{\alpha_2^{(2)}}{\omega_2}+P(\alpha_1^{(1)},\ \alpha_2^{(1)})=0,$$
$$\alpha_1^{(2)}+\alpha_2^{(2)}+Q(\alpha_1^{(1)},\ \alpha_2^{(1)})=-1.$$

Analogously are located the subsequent approach/approximations.

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By the described method, being limited to the second

approach/approximation, is obtained the value

$$J_0 \simeq 1,2.$$

For the appropriate functions u, V, it will be $\max|v'| \simeq 1, \quad \max|u| \simeq 0,5.$ We focus attention on these values because with their aid are determined the constants c' and c'' in formulas for maximum voltage/stresses (p. 2).

In connection with the study of the problem concerning elasto-plastic supercritical deformations, we will now propose another method of solving the variational problem for functional J. This solution, approximated actually, will be based on the demonstrative representations of the character of the deformation of the shell near fin/edge, with study of which is connected our variational problem.

In Fig. 3 to the right (page 26) they are depicted the section/cut of the deformed shell by the plane, perpendicular to fin/edge, and section/cut by the same plane of surface $\tilde{F}$, which approaches the form of shell. New the variables u, V, which we now use, are the respectively standardized/normalized radial displacement of the point of surface $\check{F}$ during its deformation in $\bar{F}$ and the standardized/normalized angle of rotation to tangent. The standardization of angle is carried out in such a way that its value at point A(s=+0) is equal to -1.

On the basis of the representation of the local character of the deformation of shell in the zone of powerful bending, it is logical to assume that after point B, where v=0, value V it remains small and in the differential linkage of the variables u, V term $v^2/2$ can be disregarded. Then communication/connection will take the form

$$u' + v = 0.$$

Further, it is obvious, that the maximum of the bending of the deformed shell must be reached in immediate proximity of point A. Hence it follows that near point A value V', which is determining the value of bending, changes barely, and logical to consider V' of constant in certain vicinity point A.

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Taking into account two considerations indicated, let us search for the minimum of functional J on many functions u, V, which satisfy the conditions

$$1.^{(1)}\ \text{При}\ s \leqslant \sigma \quad u' + v + \frac{v^2}{2} = 0, \quad v' = \text{const}.$$
$$2.^{(1)}\ \text{При}\ s \geqslant \sigma \qquad u' + v = 0.$$

Key: with.

Here σ - parameter, which is subject to variation. The minimum of functional J with given one σ will be known function from σ: $J_{\min} = J(\sigma)$. For determining value J_0, we minimize this function on σ:

$$J_0 = \min_{(\sigma)} J(\sigma).$$

Let us find function $J(\sigma)$. Set/assuming with $s\leqslant\sigma$

$$v'=\frac{1}{\lambda}=\text{const},$$

after integration we will obtain

$$v=\frac{s}{\lambda}+\text{const}.$$

Since v=-1 with s=0, then

$$v=\frac{s}{\lambda}-1.$$

The parameter λ makes simple sense. Specifically,, such this value s, at which V turns into zero, i.e., $\lambda=\sigma$. Thus, with $s\leqslant\sigma$ we have

$$v(s)=\frac{s-\sigma}{\sigma}.$$

From the equation

$$u'+v+\frac{v^2}{2}=0$$

we find function u(s) with $s\leqslant\sigma$:

$$u=-\frac{1}{2\sigma}(s-\sigma)^2-\frac{1}{6\sigma^2}(s-\sigma)^3+\text{const}.$$

Integration constant is determined by boundary condition u(0)=0, and it is equal to $\sigma/3$. Thus, with $s\leqslant\sigma$ it will be

$$u=-\frac{1}{2\sigma}(s-\sigma)^2-\frac{1}{6\sigma^2}(s-\sigma)^3+\frac{\sigma}{3}.$$

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The values of functions u, V at the end σ of cut $(0, \sigma)$ are respectively equal to $\sigma/3$ and 0 they are initial values for the varied functions u, V on the remaining part of the semi-axis (σ, ∞).

Since with $s \geqslant \sigma$ by hypothesis

$$u' + v = 0,$$

that functional J can be presented in the form

$$J = \int_0^\sigma (v'^2 + u^2)\, ds + \int_\sigma^\infty (u''^2 + u^2)\, ds.$$

function u(s), that realizes the minimum of functional on semi-axis (σ, ∞), it satisfies the equation of Euler

$$u^{IV} + u = 0.$$

Its general solution, which disappears at infinity, allow/assumes the representation

$$u = c_1 e^{\omega_1 s} + c_2 e^{\omega_2 s},$$

where ω_1 and ω_2 - roots of the characteristic equation

$$\omega^4 + 1 = 0$$

with negative real part, i.e.,

$$\omega_1 = -\frac{1}{\sqrt{2}}(1 - i), \quad \omega_2 = -\frac{1}{\sqrt{2}}(1 + i).$$

The constants c_1 and c_2 are determined by the conditions of the coupling of functions u, V with $s=\sigma$. We have

$$u(\sigma) = c_1 e^{\omega_1 \sigma} + c_2 e^{\omega_2 \sigma} = \frac{\sigma}{3},$$

$$v(\sigma) = -u'(\sigma) = -c_1 \omega_1 e^{\omega_1 \sigma} - c_2 \omega_2 e^{\omega_2 \sigma} = 0.$$

Hence

$$c_1 = -\frac{\sigma\omega_1}{3\sqrt{2}} e^{-\omega_1 \sigma}, \quad c_2 = -\frac{\sigma\omega_2}{3\sqrt{2}} e^{-\omega_2 \sigma}.$$

Let us calculate now value of J(σ). With $s \leqslant \sigma$ it will be

$$\vartheta = \frac{s - \sigma}{\sigma},$$

$$u = -\frac{1}{2\sigma}(s - \sigma)^2 - \frac{1}{6\sigma^2}(s - \sigma)^3 + \frac{\sigma}{3}.$$

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Hence

$$\int_0^\sigma (v'^2 + u^2)\, ds = \frac{1}{\sigma} + \left(\frac{1}{20} + \frac{1}{7\cdot 36}\right)\sigma^3.$$

With $s \geqslant \sigma$ we have

$$u = -\frac{\sigma}{3\sqrt{2}}\left(\omega_1 e^{\omega_1 (s-\sigma)} + \omega_2 e^{\omega_2 (s-\sigma)}\right),$$

$$v' = -\frac{\sigma i}{3\sqrt{2}}\left(-\omega_1 e^{\omega_1 (s-\sigma)} + \omega_2 e^{\omega_2 (s-\sigma)}\right),$$

$$u^2 + v'^2 = \frac{2\sigma^2}{9} e^{-\sqrt{2}(s-\sigma)}.$$

Hence

$$\int_\sigma^\infty (u^2 + v'^2)\, ds = \frac{2\sigma^2}{9\sqrt{2}}.$$

Thus,

$$J(\sigma) = \frac{1}{\sigma} + \frac{\sqrt{2}\,\sigma^2}{9} + \left(\frac{1}{20} + \frac{1}{7\cdot 36}\right)\sigma^3.$$

Minimizing $J(\sigma)$ on σ, we find J_0:

$$J_0 = \min_{(\sigma)} J(\sigma) \simeq 1{,}15.$$

The obtained value J_0, apparently, is close to true. In any case, application/use of electronic computers for determination J_0 gave for it the value

$$J_0 = 1{,}1156,$$

moreover the first three signs in this expression were guaranteed.

Let us calculate now the values of the constants c, c' and c'', introduced in p. 2. We have

$$c = \frac{J_0}{12^{3/4}(1-\nu^2)}.$$

Set/assuming $\nu = 0{,}3$ and J_0=1.11, we find

$$c \simeq 0{,}19.$$

Constant

$$c' = \frac{12^{1/4}}{2} \max|v'|.$$

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Set/assuming max. |V'| =1, let us have

$$c' \simeq 0{,}9.$$

Constant

$$c'' = \frac{\max|u|}{12^{1/4}}.$$

With max. |u|=0.5

$$c'' \simeq 0{,}27.$$

Let us compare the values of maximum voltage/stresses σ' and σ'' respectively from bending and elongation (compression) in median surface of shell. We have (p. 2)

$$\sigma' = 0{,}9E\,\frac{\delta^{1/2}a^{3/2}}{\rho^{1/2}}, \qquad (*)$$

$$\sigma'' = 0{,}27E\,\frac{\delta^{1/2}a^{3/2}}{\rho^{1/2}}.$$

We see that the maximum voltage/stresses from bending are much more than the tensile stress (compression) in median surface. Hence it follows that calculation for the strength of shell during supercritical deformations one should conduct according to the bending stresses in the vicinity of fin/edges. These voltage/stresses are determined from formula (*).

§2. Supercritical deformations of strictly convex hulls under external pressure.

The application/use of principle A, formulated in §1, we will illustrate in this paragraph based on the example of strictly convex hulls, which are located under external pressure. We will examine two methods of the loading of the shell: by the concentrated force, normal to the surface (at the point of its application/appendix), and by uniform external pressure. The results of theoretical examination let us compare with the data of the corresponding experiments with the shells of spherical form.

1. Determination of basic values in the case of mirror bulge of low regions. The application/use of principle A during the study of the supercritical elastic states of shell assumes the determination

of the row/series of the values of the isometric transformation of median surface.

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Keeping in mind the nearest application/appendices, we will determine such values in the case of the mirror bulge of low regions.

As is known, the form of regular strictly convex surface in the sufficiently small vicinity of this point P approaches well by certain elliptical paraboloid which is called that contacting. If we accept tangential plane at point P for plane XY, and the main directions of surface at this point for reference directions, then equation of the contacted paraboloid will take the form

$$z=\frac{1}{2}(k_1x^2+k_2y^2),$$

where k_1 and k_2 - principal curvatures of surface at point P. Hence it follows that the region of the mirror bulge of shell with the center of bulge P at the low altitude of bulge 2h (sagging/deflection at point P) is assigned by the inequality

$$\frac{1}{2}(k_1x^2+k_2y^2)\leqslant h,$$

and, therefore, is ellipse with the semi-axes

$$a=\sqrt{\frac{2h}{k_1}},\quad b=\sqrt{\frac{2h}{k_2}}.$$

In connection with the determination of energy of elastic deformation with bulge by us will be necessary the expressions for

curvature curved, that limits the region of bulge, and for normal surface curvature. Let us find expressions for these values.

The boundary of bulge as ellipse with semi-axes a, b, allow/assumes the parametric assignment

$$x = a\cos t, \quad y = b\sin t.$$

Using formula for the curvature of curve, we find

$$\frac{1}{\rho} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}},$$

where ρ - a radius of curvature.

Let us determine normal surface curvature in the direction of the boundary of bulge.

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In view of the fact that the region of bulge is small, it is possible to count that the principal curvatures on the boundary of bulge are close to k_1 and k_2 - to principal curvatures in P (center of bulge), but main directions are close to main directions into P.

On the Euler formula, normal curvature in the direction which forms angle ϑ with the main direction, which corresponds to curvature k_1, is equal to

$$k_\vartheta = k_1\cos^2\vartheta + k_2\sin^2\vartheta.$$

In the case

$$\cos\vartheta=\frac{-a\sin t}{(a^2\sin^2 t+b^2\cos^2 t)^{1/2}},\quad \sin\vartheta=\frac{b\cos t}{(a^2\sin^2 t+b^2\cos^2 t)^{1/2}}.$$

in question. Therefore normal surface curvature in the direction of the boundary of bulge is equal to

$$k_n=k_1\frac{a^2\sin^2 t}{a^2\sin^2 t+b^2\cos^2 t}+k_2\frac{b^2\cos^2 t}{a^2\sin^2 t+b^2\cos^2 t},$$

or, noting that

$$k_1=\frac{2h}{a^2},\qquad k_2=\frac{2h}{b^2},$$

we will obtain

$$k_n=\frac{2h}{a^2\sin^2 t+b^2\cos^2 t}.$$

Normal curvature in the direction, perpendicular to the boundary of bulge, is equal to

$$\bar{k}_n=k_1\sin^2\vartheta+k_2\cos^2\vartheta.$$

or, taking into account of expression for $\cos\vartheta$, $\sin\vartheta$, k_1 and k_2, we will obtain

$$\bar{k}_n=2h\frac{\frac{a^2}{b^2}\sin^2 t+\frac{b^2}{a^2}\cos^2 t}{a^2\sin^2 t+b^2\cos^2 t}.$$

Let us determine angle α between the plane curve γ, that limits the region of bulge, and by the tangential planes of surface. On formula it is less

$$\rho k_n=\sin\alpha.$$

For the low regions of bulge, and therefore small ones α, we have

$$\alpha=\rho k_n.$$

Substituting here the obtained values ρ and k_n, we will obtain

$$\alpha=\frac{2h}{ab}(a^2\sin^2 t+b^2\cos^2 t)^{1/2}.$$

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Let us calculate now U - strain energy of shell. It consists of two parts - U_G and U_γ, where U_G - energy of bending over basic surface, and U_γ - strain energy in the vicinity of the boundary γ of the region of bulge. Value U_G is determined from formula (page 25)

$$U_G = \frac{D}{2} \int \int (\Delta k_1^2 + \Delta k_2^2 + 2\nu \Delta k_1 \Delta k_2) \, d\sigma.$$

Here Δk_1 and Δk_2 - main changes in the normal curvatures upon transfer from initial form to isometric transformation,

$$D = \frac{E\delta^3}{12(1-\nu^2)}$$

- the flexural rigidity of shell, but integration is fulfilled by the area of an entire surface. In the case of the mirror bulge of value Δk_1 and Δk_2 in question out of the region of bulge G, they are equal to zero, but within this region $\Delta k_1 = 2k_1$, $\Delta k_2 = 2k_2$, where k_1 and k_2 - principal curvatures. In view of assumption about the smallness of the region of bulge, k_1 and k_2 it is possible to consider it equal to their values in the center of bulge P. Taking into account the value of the area of the region of bulge, we will obtain

$$U_G = \frac{\pi h}{3\sqrt{k_1 k_2}} \frac{E\delta^3}{(1-\nu^2)} (k_1^2 + k_2^2 + 2\nu k_1 k_2).$$

Here 2h - height/altitude of bulge (normal sagging/deflection in the center of bulge P), k_1 and k_2 - principal curvatures in P, δ - the thickness of shell, E - the modulus of elasticity, and ν - Poisson

ratio.

Let us calculate now strain energy on γ - to boundary of bulge. For it in §1 is obtained the formula (page 33)

$$U_\gamma = cE\delta^{5/2} \int_\gamma \frac{\alpha^{5/2}}{\rho^{1/2}} ds_\gamma + \frac{E\delta^3}{12(1-\nu^2)} \int_\gamma \alpha(-2k + k_e + k_i)\, ds_\gamma.$$

Here α - the angle between the plane curved γ - and the tangential planes of deformed surface, k - normal curvature of initial surface in the direction, perpendicular to the boundary of bulge, k_e and k_i - normal curvatures of the isometrically converted surface in accordance with from the side of the internal and external half-neighborhood of the boundary of bulge.

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In the case of mirror bulge $k_e = k, \ k_i = -k.$ Thus, second term in formula for U_γ can be written in the form $-\frac{E\delta^3}{6(1-\nu^2)} \int_\gamma k\alpha\, ds_\gamma.$ Taking into account the obtained above values for normal curvature $k = \bar{k}_n$, angle α and noting that $ds_\gamma = (a^2 \sin^2 t + b^2 \cos^2 t)^{1/2} dt,$ let us have

$$\int_\gamma k\alpha\, ds_\gamma = \frac{4h^2}{ab} \int_0^{2\pi} \left(\frac{a^2}{b^2} \sin^2 t + \frac{b^2}{a^2} \cos^2 t\right) dt = \\ = \frac{4\pi h^2}{ab} \left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right).$$

Thus, second term in expression for U_γ is equal

$$-\frac{E\delta^3 \pi h}{3(1-\nu^2)} (k_1^2 + k_2^2).$$

When $\nu=0$ it differs from expression U_G only in terms of sign.

We assume that the more in-depth analysis of the elastic state of shell at the boundary of bulge must bring to by complete the compensation this term energy U_G also, when $\nu \neq 0$. Therefore total energy of deformation let us determine from the formula

$$U = cE\delta^{5/2} \int_{\gamma} \frac{\alpha^{5/2}}{\rho^{1/2}}\, ds_{\gamma}.$$

Substituting in this formula the obtained above values α and ρ, we will obtain

$$\int_{\gamma} \frac{\alpha^{5/2}}{\rho^{1/2}}\, ds_{\gamma} = \frac{(2h)^{5/2}}{a^2 b^2} \int_{0}^{2\pi} (a^2 \sin^2 t + b^2 \cos^2 t)\, dt =$$

$$= \frac{(2h)^{5/2}}{a^2 b^2} \pi (a^2 + b^2) = \pi (2h)^{5/2} (k_1 + k_2).$$

Hence

$$U = \pi c E \delta^{5/2} (2h)^{5/2} (k_1 + k_2).$$

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Let us estimate magnitude of error which we allow/assume when $\nu \neq 0$, reject/throwing in the expression of strain energy term/component/addend

$$\Delta U = \frac{\pi h}{3\sqrt{k_1 k_2}} \frac{E\delta^3}{1-\nu^2} (2\nu k_1 k_2).$$

Let us take for an example the spherical shell of radius R. For this shell the found by us value of strain energy is equal

$$U = 2\pi c E \delta^{5/2} (2h)^{5/2} \frac{1}{R},$$

the reject/thrown term is equal to

$$\Delta U=\frac{\pi h}{3}\frac{E\delta^3}{1-\nu^2}2\nu\frac{1}{R},$$

and relative error is equal to

$$\frac{\Delta U}{U}=\frac{\nu}{6(1-\nu^2)c}\sqrt{\frac{\delta}{2h}}.$$

Since we examine the considerable deformations (2h/δ greatly), then hence it is apparent that when $\nu\neq 0$ introduced by our assumption error is small, even if this assumption is erroneous.

Let us determine the maximum voltage/stresses σ in the material of shell with bulge. Such voltage/stresses appear from bending on the boundary of bulge and are determined from formula (§1, p. 2)

$$\sigma=c'E\frac{\delta^{1/2}\alpha^{3/2}}{\rho^{1/2}}.$$

Substituting here values α and ρ, we will obtain

$$\sigma=c'E\delta^{1/2}(2h)^{1/2}\sqrt{k_1k_2}.$$

It is substantial to note that these voltage/stresses are constant along the boundary of bulge.

Thus, with the mirror bulge of the low region of strictly convex hull strain energy U is determined from the formula

$$U=\pi cE\delta^{5/2}(2h)^{3/2}(k_1+k_2),\quad c\simeq 0{,}19.$$

The maximum, appearing from bending on boundary bulges of voltage/stress are equal to

$$\sigma=c'E\delta^{1/2}(2h)^{1/2}\sqrt{k_1k_2},\quad c'\simeq 0{,}9,$$

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2. Supercritical deformations of shells under action of concentrated force. Let the strictly convex hull, rigidly attached on edge, locate under the action of concentrated force of f, normal to the surface of shell at the point of application/appendix. If this force causes considerable deformation, then the determination of the elastic state of shell is reduced to task to the extremum of functional W=U-A which is determined and is examined during the isometric transformations of the initial form of shell. We will assume that the bulge of shell, caused by the action of force of f, encompasses convex region. In this case, as shown in p. 1 §1 the class of the isometric transformations during which it is necessary to examine our variational problem, becomes narrow to mirror bulge.

In the case of mirror bulge for functional U in p. 1 obtained following expression:

$$U = \pi c E \delta^{5/2} (2h)^{5/2} (k_1 + k_2).$$

Assuming that the point of the application of force of f is the center of bulge, let us have for functional A, which is the work, produced by force of f by the deformation of shell, formula A=2fh.

From the condition of stability d(U-A)=0 of functional W, we obtain the dependence between the amount of acting force f and deformation (2h), which it causes. We have

$$dW = 3\pi c E\delta^{5/2}(2h)^{1/2}(k_1 + k_2)\,dh - 2f\,dh = 0.$$

Hence

$$f = \frac{3\pi c}{2} E\delta^{5/2}(k_1 + k_2)\sqrt{2h}.$$

From this formula it is evident that during an increase in the deformation the received by shell load f increases. This indicates the stability of the states of the equilibrium of shell under the action of concentrated load.

To conclusion/derivation about the stability of the states of equilibrium it is possible to proceed by another way, examining the second variation in functional W. We have

$$d^2W = 3\pi c E\delta^{5/2}(k_1 + k_2)\frac{1}{\sqrt{2h}}\,dh^2 > 0.$$

But this means that the state of equilibrium is stable.

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Let us examine especially the case of spherical shell. For the spherical shell of radius R, we have $k_1 = k_2 = 1/R$, and the formula, establishing the dependence between the acting force f and the

sagging/deflection 2h, which it causes, takes the form

$$f = 3\pi c E \delta^{5/2} \sqrt{2h} \frac{1}{R}.$$

If a radius of the circle of bulge is designated through ρ and is replaced, that $2h \simeq \rho^2/R$, then this dependence can be still written then

$$f = 3\pi c E \frac{\delta^{5/2}}{R^{3/2}} \rho.$$

Thus, the dependence of a radius of the circle of bulge on the acting force f is linear.

The obtained dependence of sagging/deflection 2h on the acting force f for spherical shells was subjected to experimental check[1].

FOOTNOTE [1]. These experiments, just as others, described below, are carried out by the author with the participation of the colleagues: M. M. Pugolovok, Yu. i. Kravetskiy, N. S. Sukhlenk, N. I. Kotov, V. A. Chistyukhin, A. N. Pedorenko. Experiments were conducted in the physiotechnical low-temperature institute of AS UkSSR. ENDFOOTNOTE.

The installation during which was conducted the corresponding experiment, was arranged sufficiently simply and it is schematically represented in Fig. 4.

Tested spherical segment 1 freely rests on rigid ring by 2. The action of load f, which consists of the calibrated according to weight washers, through vertical rod 3 is transferred to the surface of segment.

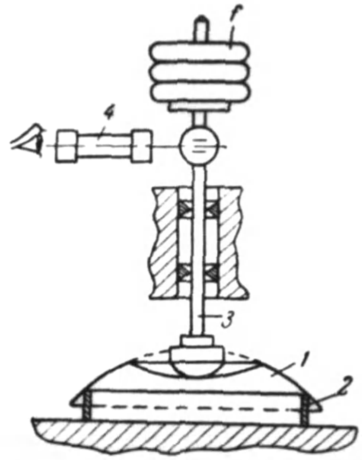

Fig. 4.

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In order to exclude the plastic deformations of segment in immediate proximity of the point of the application/appendix of concentrated force, the point of rod, which contacts with the surface of segment, is carried out with comparatively small, but larger than in segment, by curvature. The vertical displacement/movements of rod, i.e., the sagging/deflections of shell (2h), were recorded with the aid of precise optical instrument by 4, making it possible to measure these displacement/movements with an accuracy to 10^{-3} mm.

Experiment was conducted on the series of the copper shells of radius R=150 mm with different thickness δ from 0.03 to 0.10 mm.

Shells were obtained by metal spraying copper in vacuum to the steel backing of spherical form. The sphericity of backing and, consequently, also the obtained shells, was maintained with high (optical) accuracy/precision. The special conditions/modes of metal spraying made it possible to obtain the specimen/samples, possessing high elastic limit. This is substantial for experiments with supercritical deformations, since the voltage/stresses in the zone of powerful local bending (on the boundary of bulge) during such deformations are very considerable.

Figure 5 depicts the graph/diagrams of the theoretical dependence of the sagging/deflections of shell under the action of the concentrated force

$$2h=\frac{R^2}{9\pi^2c^2E^2\delta^5}f^2$$

for the spherical shells of radius R=150 mm and different thicknesses δ=0.037, 0.048 and 0.056 mm. Graphs are constructed taking into account the actual value of the module/modulus of elasticity E which was determined by special testing for bending of the flat/plane specimen/samples, obtained by metal spraying under similar conditions.

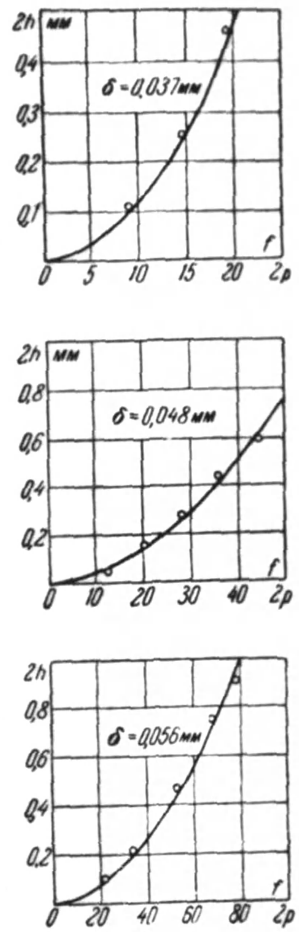

Fig. 5.

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Isolated points on graphs give the experimental values of sagging/deflections 2h at the different values of the acting force f. We see that the theory and experiment in a question in question are in satisfactory agreement.

Theoretical and experimental study of a question concerning the deformations of strictly convex hull under the action of concentrated force we is summed up by following conclusion.

The sagging/deflection 2h of strictly convex hull under the action of concentrated force of f, normal to the surface of shell, it is determined by the relationship/ratio

$$f = \frac{3\pi c}{2} E\delta^{5/2}(k_1 + k_2)\sqrt{2h},$$

where k_1 and k_2 - principal curvatures of shell at the point of application of force δ - thickness of shell, and is constant $c \simeq 0.19$. In particular, for the spherical shell of radius R

$$f = 3\pi c E\delta^{5/2}\frac{\sqrt{2h}}{R}.$$

3. Supercritical deformations of strictly convex hull under external pressure. Let the strictly convex hull, rigidly attached on edge, locate under the action of the uniform external pressure p. Let us examine the states of the elastic equilibrium of the shells with which its form experience/tests supercritical deformations. According to principle A (§1, page 32) the determination of these states is reduced to the solution of task to extremum for the functional

$$W = U - A$$

on many isometric transformations of initial surface. If one assumes that the deformation is accompanied by the bulge of convex region, then the class of the isometric transformations, during which is examined the functional, it becomes narrow before the isometric

transformations, which are reduced to mirror bulge (§1, p. 1).

In the case of mirror bulge, the strain energy U is determined from formula (page 49)

$$U = \pi c E (2h)^{3/2} \delta^{5/2} (k_1 + k_2),$$

where 2h - height/altitude of bulge in the center of region, k_1 and k_2 - principal curvatures of median surface, δ - the thickness of shell, E - the module/modulus of elasticity, while constant c≃0.19.

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Let us determine that produced by the external pressure p work A. It is equal to the product of the value of pressure on a change (during deformation) in the volume, limited by shell, i.e.,

$$A = p\,\Delta V.$$

It is found ΔV. After accepting the center of bulge P in the origin of coordinates, and tangential plane in P for plane XY, in the corresponding direction of axle/axes X, y we can assign the surface of shell near P equation

$$z = \frac{1}{2}(k_1 x^2 + k_2 y^2).$$

If we designate through S(z) the area of the region, determined by the inequality

$$\frac{1}{2}(k_1 x^2 + k_2 y^2) \leqslant z,$$

then the which interests us volume

$$\Delta V = 2\int_0^h S(z)\,dz.$$

Area S(z), as the area of ellipse with the semi-axes

$$a=\sqrt{\frac{2z}{k_1}},\quad b=\sqrt{\frac{2z}{k_2}}$$

it is equal to πab, and therefore

$$S(z)=\pi\frac{2z}{\sqrt{k_1k_2}}.$$

Hence after integration for z, we obtain the volume

$$\Delta V=\frac{2\pi h^2}{\sqrt{k_1k_2}}.$$

The produced by external pressure p work is equal to

$$A=\frac{2\pi h^2 p}{\sqrt{k_1k_2}}.$$

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Now from stability condition of functional W in the state of the elastic equilibrium of shell, we find the dependence of the received by shell pressure p on sagging/deflection 2h in the center of bulge. We have

$$dW=3\pi cE\,(2h)^{1/2}\,\delta^{5/2}(k_1+k_2)\,dh-\frac{4\pi hp}{\sqrt{k_1k_2}}\,dh=0,$$

whence

$$p=\frac{3}{2}\,cE\,(k_1+k_2)\sqrt{k_1k_2}\,\frac{\delta^{5/2}}{\sqrt{2h}}.$$

From this formula it is evident that the received by shell pressure p is decreased during an increase in deformation (2h). But this indicates the instability of supercritical deformations under external pressure. Conclusion about the instability of the obtained

elastic states corresponds to the experimental data on the character of supercritical deformations under external pressure. According to these data, supercritical deformations after the loss of stability of shell are developed without an increase in the load and even during its decrease.

The smallest received by shell load during supercritical deformation is called lower critical in contrast to the upper critical load, with which occurs the loss of stability of basic form. Let us examine a question concerning the value of lower critical load for strictly convex hulls, which are located under external pressure. In view of the fact that the received by shell load is decreased during an increase in the deformation, lower critical load corresponds to the greatest geometrically permissible deformation. If this deformation is designated $2h_l$, then lower critical load p_l will be determined from the formula

$$p_l = \frac{3}{2} cE (k_1 + k_2) \sqrt{k_1 k_2} \frac{\delta^{5/2}}{\sqrt{2h_l}}.$$

Let us examine as an example the strictly convex hull of rotation, reinforced by the rigid cell/elements, which go along parallels and meridians of surface (Fig. 6).

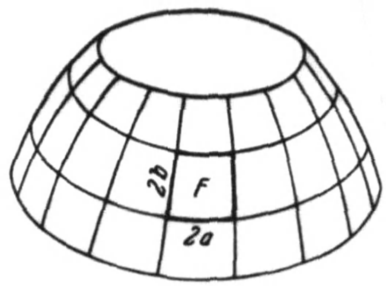

Fig. 6.

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F - section of shell, limited by the supporting cell/elements, 2a and 2b - size/dimensions of section on parallel and meridian respectively. Since the size/dimensions of the region of bulge according to main directions at the height/altitude of bulge 2h are equal to

$$\sqrt{\frac{2h}{k_1}} \text{ and } \sqrt{\frac{2h}{k_2}}.$$

that maximum deformation $2h_i$ of section F is determined smaller of two values

$$k_1a^2 \text{ and } k_2b^2.$$

where k_1 - normal surface curvature in the direction of parallel, but

k_2 - normal curvature on meridian, i.e.,

$$2h_i = \min\{k_1 a^2, k_2 b^2\}.$$

Substituting value $2h_i$ in formula for p, we will obtain the lower critical pressure

$$p_i = \frac{3}{2} cE (k_1 + k_2) \sqrt{k_1 k_2} \frac{\delta^{5/2}}{\sqrt{2h_i}}.$$

In connection with the experimental check of the obtained result about the value of lower critical load, let us examine the supercritical deformations of the rigidly attached on edge spherical segment. If the radius of curvature of segment is equal to R, then the received by it pressure p with bulge on height/altitude 2h is determined from the formula

$$p = 3cE \left(\frac{\delta}{R}\right)^2 \sqrt{\frac{\delta}{2h}}.$$

At the height/altitude of segment h_0 for the maximum geometrically permissible deformation 2h, we have

$$2h = 2h_0.$$

Therefore lower critical pressure for a spherical segment is equal

$$p_i = 3cE \left(\frac{\delta}{R}\right)^2 \sqrt{\frac{\delta}{2h_0}}.$$

If we into this formula introduce instead of height/altitude h_0 a radius of the basis/base of the segment

$$r \simeq \sqrt{2h_0 R},$$

then it takes the form

$$p_i = 3cE \left(\frac{\delta}{R}\right)^2 \sqrt{\frac{\delta R}{r^2}}.$$

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Let us explain now the region of the applicability of the obtained results to the real shells, which possess the limited elasticity. Our basic assumption in the examination of supercritical deformations consisted in the fact that a change in the form of shell during such deformations is very considerable. Virtually this means that the size/dimensions of the region of bulge are of the order of the size/dimensions of an entire shell. In view of the limited elasticity of the material of shell, its elastic deformations are naturally limited, and this limits the size/dimensions of shells, to which the obtained results are used. In order to give to these limitations concrete/specific/actual form, let us examine for an example the spherical shell in the form of segment.

The maximum voltage/stresses in the material of shell (on the boundary of bulge) during deformation 2h are equal (page 49)

$$\sigma = c'E\,(2h)^{1/2}\,\delta^{1/2}\,\frac{1}{R}.$$

If the time/temporary strength of materials of shell is designated σ_s, then the region of its elastic deformations is limited to the condition

$$c'E\,(2h)^{1/2}\,\delta^{1/2}\,\frac{1}{R} < \sigma_s,$$

or, by introducing instead of 2h a radius of the circle of bulge ρ on the formula

$$2h = \frac{\rho^2}{R},$$

we will obtain

$$c'E\frac{\rho}{R}\sqrt{\frac{\delta}{R}}<\sigma_{\theta}.$$

Hence it follows that

$$\frac{\rho}{R}<\frac{\sigma_{\theta}}{c'E}\sqrt{\frac{R}{\delta}}.$$

In order deformation to consider considerable, it is necessary that ρ would be of the order of a radius of the basis/base of segment r. Thus, our the order of a radius of the basis/base of segment r. Thus, our examinations are related to such spherical shells whose values

$$\frac{r}{R} \text{ and } \frac{\sigma_{\theta}}{c'E}\sqrt{\frac{R}{\delta}}$$

have one order.

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During the usual relationship/ratios between values σ_{θ}, E, 6 and R this it means that the shells must be very flat.

To an even more rigorous condition is limited the application/use of a formula for a lower critical load. Specifically,, since lower critical load corresponds to the maximum geometrically permissible deformation, the corresponding condition

for spherical shells is reduced to the fact that

$$\frac{r}{R} \leqslant \frac{\sigma_s}{c'E} \sqrt{\frac{R}{\delta}}.$$

Lower critical load for flat spherical segments was subjected to experimental study. The corresponding experiment consisted of following.

Inside massive cylindrical container 1 (Fig. 7), closed from above by tested spherical shell 2, with the aid of micrometer gauge was supplied piston with 3. In this case, in the liquid, which fills container, built up the pressure, which was recorded by the specific equipment/device. After achieving critical value, pressure began to be decreased and it descended to certain minimum, after which again it increased. Maximum pressure answers the torque/moment of loss of stability of shell and is upper critical pressure. But the minimum of pressure corresponds to lower critical load, as we it determined.

Let us note some design features of experimental installation and work on it. First of all, we attempted to avoid the sharp "cotton/knock", by which is usually accompanied the loss of stability of shell in the experiments of this type. In connection with this all elastic elements of construction/design were made "maximally rigid". For this very reason as the medium, which fills container and which communicates pressure on shell, was undertaken liquid, but container itself was carried out sufficiently to massive ones, with thick walls.

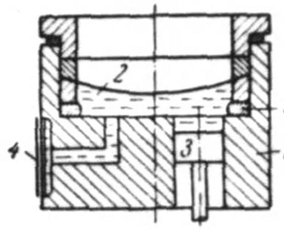

Fig. 7.

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Pressure was recorded with the aid of very sensitive strain gauge by 4, fastened/strengthened to surface container. As a result of all measures indicated the pressure as a result of "cotton/knock" did not descend to the minimum and it reached this minimum only with further movement of piston inside container. This is important for the shells with the limited elasticity, which after energetic "cotton/knock" can show even negative lower critical pressure.

Tested segment was stopped up between two steel rings of which it lower rested on rubber packing by 5, but it was upper pressed by flange. The conditions of the jamming of segment for edge, close to ideal ones, were provided by the grinding of rings over the appropriate spherical surfaces and by the uniformity of pressure rings because of the elasticity of packing 5.

Four strain gauges 4, arrange/located on the lateral surface of container, two in circumference, and two others in axial direction - were connected in the usual way into bridge circuit which was connected to supply of power and galvanometer. Readings of galvanometer preliminarily were calibrated.

Testing underwent the copper spherical segments, obtained by metal spraying in vacuum. The radius of curvature of segments R=80 mm, and thickness δ vary within the range of 0.03 to 0.09 mm. The bore of the rings 2r, which clamp tested segments, was equal to 2r=16 by mm.

Figure 8 broken lines depicts the dependence of lower critical pressure p_l, given by the formula

$$p_l = 3cE\left(\frac{\delta}{R}\right)^2 \sqrt{\frac{\delta R}{r^2}},$$

on the thickness of shell δ. The module/modulus of elasticity E is accepted equal to $1.1 \cdot 10^6$ kg/cm², the constant c=0.19, but R and r have values indicated above. The isolated points, noted by circles, give the values of lower critical pressure, obtained in experiment.

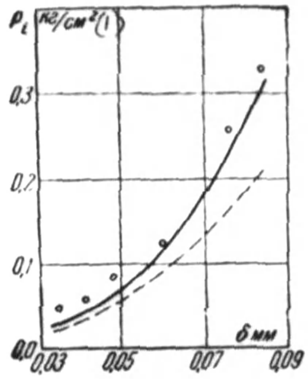

Fig. 8.

Key: (1). Kg/cm².

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We see that for all values δ theoretical value p_i is lower than the experimental. And this is understandable. Really/actually, during the derivation of formula for p_i we considered that during the deformation, corresponding to lower critical pressure, a radius of the circle of bulge ρ was equal to a radius of the basis/base of segment r. In actuality always

$$\rho < r.$$

Therefore for obtaining the true value of lower critical pressure, it is necessary into formula for p_i to substitute for r somewhat smaller value. To what extent it is smaller, we now will explain.

Figure 9 depicts the section/cut of spherical shell during

supercritical deformation. The received by shell load p and deformation (ρ) are connected by relationship/ratio (page 56)

$$p = 3cE\left(\frac{\delta}{R}\right)^2 \sqrt{\frac{\delta R}{\rho^2}}.$$

where ρ - a radius of the circle of bulge. A - point on the boundary of bulge, B - the nearest to A point at which radial displacement u, caused by the deformation in question, is equal to zero. If at point B there was $v'=0$, then along parallel γ_B, passing through point B, would be realized the condition of rigid attachment. The formula

$$p = 3cE\left(\frac{\delta}{R}\right)^2 \sqrt{\frac{\delta R}{\rho_A^2}} \qquad (*)$$

would give the value of lower critical pressure for the segment, limited by parallel γ_B, under the condition of rigid attachment along this parallel. However, at point B, condition $v'=0$ is not satisfied. Therefore formula (*) gives the critical pressure, which corresponds only to elastic attachment of shell along parallel γ_B (elasticity in the rotation of tangential planes). If the segment, limited by parallel γ_B, is identified with subject, then corresponding to it pressure p_i, determined on formula (*), still will be less than the true, but already it is much nearer to it. Let us calculate this refined value p_i.

Fig. 9.

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ρ - radius of the parallel, passing through point A, and r - radius of the parallel, passing through B. Let us assume

$$\Delta\rho = r - \rho.$$

If we pass to dimensionless quantities as this was done in §1, p. 2, then nondimensional distance $\overline{\Delta\rho}$ between points A and B will be the first different from zero roots of the equation

$$u(s) = 0,$$

where u(s) - the function, which realizes the minimum of functional J (§1, p. 2). Taking into account the explicit expression of the function

$$u(s) = -\frac{\sigma}{3\sqrt{2}}\left(\omega_1 e^{\omega_1(s-\sigma)} + \omega_2 e^{\omega_2(s-\sigma)}\right)$$

and of the value $\sigma \simeq 1{,}25,\quad \omega_1 = \frac{1}{\sqrt{2}}(-1+i),\quad \omega_2 = \frac{1}{\sqrt{2}}(-1-i),$

we find

$$\overline{\Delta\rho} = \sigma + \frac{\pi}{2\sqrt{2}} \simeq 2{,}36.$$

In value $\overline{\Delta\rho}$, it is possible to find value $\Delta\rho$, since they are connected by the relationship/ratio

$$\overline{\Delta\rho} = \frac{\Delta\rho}{\rho\varepsilon},\quad \varepsilon^4 = \frac{\delta^2}{12\rho^2 a^2}.$$

Hence

$$\frac{\Delta\rho}{\rho}=\Delta\bar{\rho}\sqrt{\frac{\delta}{\rho\alpha}}\frac{1}{12^{1/4}}.$$

Substituting here $\alpha\simeq\rho/R$, we find

$$\Delta\rho=\frac{\Delta\bar{\rho}}{12^{1/4}}\sqrt{\delta R}\simeq 1,3\sqrt{\delta R},$$
$$\rho=r-\Delta\rho=r-1,3\sqrt{\delta R}.$$

Thus, for a lower critical value p_l occurs the following refined formula:

$$p_l=\frac{3cE\left(\frac{\delta}{R}\right)^2\sqrt{\delta R}}{r-1,3\sqrt{\delta R}}.$$

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The graphic representation of this dependence Fig. 8 depicts by solid line. We see that the refined value p_l lower than is as before the true, but it is considerably nearer to it.

The conducted investigation of the supercritical deformations of strictly convex hulls under external pressure can be summed up as follows:

1. During the supercritical deformation flat rigidly convex hull, rigidly attached on edge, the received by it load p (external pressure) depending on sagging 2h in the center of bulge is determined from the **formula**

$$p=\frac{3}{2}cE(k_1+k_2)\sqrt{k_1k_2}\frac{\delta^{5/2}}{\sqrt{2h}}.$$

2. Lower critical **load** p_l. **i.e., the smallest** received by shell load, is determined by the maximum geometrically permissible deformation $2h_l$ **from the formula**

$$p_l=\frac{3}{2}cE(k_1+k_2)\sqrt{k_1k_2}\frac{\delta^{5/2}}{\sqrt{2h_l}}.$$

Determined by th**is formula value** p_l is lower than the true, but it is close to it)[1].

FOOTNOTE [1]. It goes without saying, when deformation 2h approaches geometrically permissible $2h_l$, begins to manifest itself the attachment of edge. So that the obtained formula, strictly speaking, gives lower limit for a lower critical load. Given by formula value of lower critical load is more precise, the less the ratio/relation $\delta/2h_l$. ENDFOOTNOTE.

For spherical segments occurs the refined formula of lower critical load. Specifically,,

$$p_l=\frac{3cE\left(\frac{\delta}{R}\right)^2\sqrt{\delta R}}{r-1{,}3\sqrt{\delta R}}.$$

3. All enumerated results are used for the real shells, which possess the limited elasticity, only with them sufficient on the plane. In particular, the application/use of a formula for lower critical pressure in the case of spherical segments assumes made the condition

$$\frac{r}{R}<\frac{\sigma_{\theta}}{c'E}\sqrt{\frac{R}{\delta}}.$$

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Examining a question concerning lower critical load for common/general/total strictly convex hulls, we assumed their sufficient flatness. The condition of flatness consisted in the fact that on the region of bulge the tangential planes of median surface of shell formed small angles, and normal curvatures differed little from some average/mean values. This made it possible to solve task in the closed form for the shells of arbitrary form. However, in each specific case method which we used makes it possible to solve problem, also, with more common/general/total propositions when, in particular, the second condition, which relates to normal curvatures, can and not be fulfilled. As an example let us find external lower

critical pressure for the ellipsoidal bottom of cylindrical reservoir.

We will assume that the supercritical deformation of bottom possesses axial symmetry as initial form. The strain energy of bottom is equal to

$$U = 2\pi c E \delta^{5/2} \alpha^{5/2} \rho^{1/2},$$

where δ - the thickness of bottom, ρ - a radius of the region of bulge, α - an angle between the plane curved, that limits the region of bulge, and by the tangential planes of surface, E - a module/modulus of elasticity, and the constant $c \simeq 0.19$.

The produced by the external pressure p work by the bulge of bottom is equal to

$$A = pV,$$

where V - a change in the volume of reservoir during the deformation of bottom.

Let us characterize the bulge of bottom the parameter ρ. Then the condition of the elastic equilibrium of bottom with bulge will be

$$\frac{d}{d\rho}(U - A) = 0.$$

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Assuming sufficient flatness of the region of bulge, let us have

$$\frac{d\alpha}{d\rho} \simeq k,$$

where k - curvature of bottom in initial form according to radial section. Hence

$$\frac{dU}{d\rho}=\pi c E \delta^{5/2} \alpha^{5/2} \rho^{-1/2}\left(5\,\frac{\rho}{\alpha}\,k+1\right).$$

Let us designate through h the height/altitude of the mirror reflected segment. Then

$$\frac{dA}{d\rho}=p\,\frac{dV}{dh}\,\frac{dh}{d\rho},\quad \frac{dV}{dh}=2\pi\rho^2,\quad \frac{dh}{d\rho}\simeq\alpha.$$

Thus,

$$\frac{dA}{d\rho}=2\pi p\rho^2\alpha.$$

Substituting the obtained values $dU/d\rho$ and $dA/d\rho$ in the equation of equilibrium, we obtain communication/connection between the received pressure p and the deformation of the bottom:

$$p=\frac{1}{2}\,cE\left(\frac{\delta}{\rho}\right)^{5/2}\alpha^{3/2}\left(5\,\frac{\rho}{\alpha}\,k+1\right).$$

Let us introduce instead of ρ the parameter $\xi=\rho/R$, where R - a radius of the basis/base of bottom. Depending on this parameter of value α and k, they are expressed on the formulas

$$\alpha\simeq\operatorname{tg}\alpha=\frac{\lambda\xi}{(1-\xi^2)^{1/2}},\quad k\simeq\frac{\lambda}{R}\,\frac{1}{(1-\xi^2)^{3/2}},$$

where λ - ratio of the height/altitude of bottom to a radius of basis/base, i.e., the ratio/relation to the semiminor axis of ellipsoid to semimajor axis. Substituting the obtained values α and k in dependence on ξ into formula for p, we will obtain

$$p=\frac{c}{2}\,E\left(\frac{\delta}{R}\right)^{5/2}\lambda^{3/2}\theta(\xi),$$
$$\theta(\xi)=\frac{1}{\xi}\,\frac{1}{(1-\xi^2)^{3/4}}\left(\frac{5}{1-\xi^2}+1\right).$$

Lower critical pressure p_l it answers the smallest value θ. It is obtained with $\xi \simeq 0.5$ and it is equal $\simeq 18.8$. Hence, taking into account value the constant $c \simeq 0.19$, we obtain the following formula for the lower critical pressure

$$p_l = 1{,}8\,E\left(\frac{\delta}{R}\right)^{5/2}\lambda^{4/3}.$$

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In the course of our conclusion/derivation, we previously assumed that α is small. Let us show that this assumption is fulfilled, if is sufficiently small λ. Really/actually

$$\alpha \simeq \frac{\lambda\xi}{(1-\xi^2)^{1/2}}.$$

With $\xi = 0.5$

$$\alpha \simeq 0{,}5\,\lambda,$$

and, therefore, α is small together with λ. It is possible to count that the condition of smallness α is satisfied, if $\lambda < 0.5$.

#

4. Elasto-plastic supercritical deformations. Supercritical deformation, being it is connected with considerable changes in exterior form of shell, leads to the very large voltage/stresses in

the material of shell, in particular on the boundary of the region of bulge. Therefore the real shells, which possess the limited elasticity, as a rule, experience/test in this case elasto-plastic deformation. In connection with this is of unconditional interest the investigation of a question concerning how occur/flow/lasts the supercritical deformation of shells with the limited elasticity.

Let us assume the supercritical deformation of shell is so considerable that the voltage/stresses from curvature on the boundary of bulge cause plastic deformations. Let us calculate energy of the elastoplastic deformation of the cell/element of shell on the boundary of bulge.

We will assume that the material of shell has classical constitution diagram. This means that the relative deformation (elongation - compression) ε and its calling voltage/stresses σ are connected by the dependence, presented in Fig. 10. Thus, when $|\varepsilon| \leqslant \varepsilon_e$ deformation is elastic, and the corresponding to it voltage/stresses σ in material are determined from the formula

$$\sigma = \varepsilon E.$$

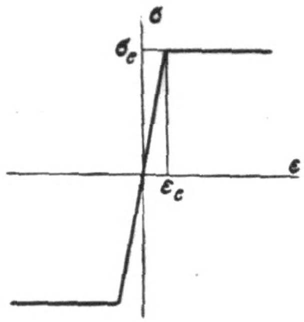

Fig. 10.

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Deformation ε when $|\varepsilon|>\varepsilon_e$ is plastic. In the plastic flow area of voltage/stress in material, they remain the constants, equal to σ_e with elongation and $(-\sigma_e)$ during compression.

Let the cell/element of shell undergo considerable curvature. If the change in the curvature of median surface of shell, caused by this curvature, is equal to k, then relative tensile strain (compression) in the material of shell at a distance of h from median surface will be

$$\varepsilon=kh.$$

In this case, if k is great so that $\varepsilon\gg\varepsilon_e$, then the strain energy, in reference to unit volume, on this distance will be

$$A(h)\simeq\sigma_e kh.$$

The strain energy of shell, in reference to the unit of area of

median surface, is equal to

$$A=2\int_{0}^{\delta/2}A(h)\,dh=\sigma_e k\frac{\delta^2}{4}.$$

Let us assume now that the cell/element of shell experience/tests considerable curvature with a change in the curvature on k first in one direction, and then in opposite with the restoration/reduction of initial form. Energy of this deformation of shell per the unit of area of median surface will be

$$A'=\frac{\sigma_e k\delta^2}{2}.$$

Let in the course of the supercritical deformation of shell the region of bulge be expanded also at certain torque/moment on its boundary appear plastic deformations. Let us calculate energy of elasto-plastic deformation in the external half-neighborhood of the conditional fin/edge, which limits the region of bulge. We will assume that the plastic deformations of shell appear only from curvature in the plane, perpendicular to fin/edge. The deformation of median surface is assumed to be elastic.

Let us designate through ε_e^* the width of the external half-neighborhood of fin/edge, encompassed by plastic deformations.

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Fide *

Total energy of deformation U in external half-neighborhood can be presented in the form

$$U = U' + U''.$$

where U' - energy of elasto-plastic deformation in immediate proximity of fin/edge, U" - energy of purely elastic deformation in the remaining part of the zone of powerful bending.

Retaining designations p. 2 of §1, we can write

$$\bar{U}'' = \frac{\delta E}{1-\nu^2}\int_{\varepsilon_e^*}^{\varepsilon^*}\left(\frac{\delta^2 v''^2}{12} + \frac{u^2}{\rho^2}\right)ds.$$

Energy of elasto-plastic deformation is equal to

$$U' = \frac{\delta E}{1-\nu^2}\int_0^{\varepsilon_e^*}\frac{u^2}{\rho^2}\,ds + \int_0^{\varepsilon_e^*}\frac{\sigma_e v''\delta^2}{4}\,ds$$

or

$$\bar{U}' = \frac{\delta E}{1-\nu^2}\int_0^{\varepsilon_e^*}\frac{u^2}{\rho^2}\,ds + \frac{\sigma_e\delta^2}{4}\left(v'(\varepsilon_e^*) - v'(0)\right).$$

Here the second term considers energy of elasto-plastic deformation from bending in the plane, perpendicular to fin/edge, and the first

term - strain energy in median surface, which accompanies this bending. Both of formulas give the strain energy, in reference to the unit of the length of fin/edge.

So as in §1 during the investigation of elastic deformations, let us introduce together the variables u, v, s new the variables $\bar{u}$, $\bar{v}, \bar{s}$ according to the formulas

$$\bar{u}=\frac{u}{\varepsilon\rho\alpha^2}, \qquad \bar{v}=\frac{v'}{\alpha}, \qquad \bar{s}=\frac{s}{\rho\varepsilon}, \qquad \varepsilon^4=\frac{\delta^2}{12\rho^2\alpha^2}.$$

In new variables above designations of which the feature lowers, let us have

$$U''=K\int\limits_{\bar{\varepsilon}_e^*}^{\bar{\varepsilon}^*}(v'^2+u^2)\,ds,$$

$$\bar{U}'=K\int\limits_{0}^{\bar{\varepsilon}_e^*}u^2\,ds+\frac{\alpha\delta^2\sigma_e}{4}\left(v(\bar{\varepsilon}_e^*)+1\right),$$

where $\bar{\varepsilon}_e^*$ and $\bar{\varepsilon}^*$ - new integration limits.

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Under the same assumptions that and in §1, integration limit $\bar{\varepsilon}^*$ can be taken as equal to ∞. If we in this case $\bar{\varepsilon}_e^*$ for simplicity of recording designate σ, then total energy $\bar{U}$ of elasto-plastic

deformation in external half-neighborhood of rib it is possible to write then:

$$\bar{U}=K\int_{0}^{\infty}(\vartheta v'^{2}+u^{2})\,ds+\frac{\alpha\delta^{2}\sigma_{e}}{4}(v(\sigma)+1),$$

where $\vartheta(s)=0$ with $s\leqslant\sigma$, $\vartheta(s)=1$ with $s>\sigma$,

$$K=\frac{E\delta^{5/2}a^{5/2}}{2\cdot 12^{3/4}\rho^{1/2}}.$$

The true form which accepts the shell in the external half-neighborhood of fin/edge - the boundary of bulge - it is determined from the condition of the minimum of functional $\bar{U}$. Let us examine the task of the minimum of this functional.

On the basis of the demonstrative representations of the character of the deformation of shell in the external half-neighborhood of fin/edge, let us assume that the plastic flow area from bending encompasses section AB (Fig. 11). Point B is determined by that condition that the tangent in it is parallel to external semi-tangent in fin/edge. Analytically the position of point B is determined by the condition

$$V(\sigma)=0.$$

Under this assumption relative to the zone plastic deformations for

strain energy $\bar{U}$ let us have

$$\bar{U}=K\int_{0}^{\infty}(\vartheta v'^{2}+u^{2})ds+\frac{\alpha\delta^{2}\sigma_{e}}{4}.$$

Just as in the case of unlimitedly elastic shells (§1), it will consider that v'=const with s≤σ.

Fig. 11.

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Further, assuming by v by comparatively small with $s>\sigma$, let us drop/omit term $v^2/2$ in the differential linkage of functions u, v

$$u' + v + \frac{v^2}{2} = 0.$$

Record/fixing σ, let us find the minimums of functional $\bar{U}$ and functions u, v, which it realize.

Since v'=const with $s \leqslant \sigma$, and

$$v(0) = -1, \quad v(\sigma) = 0,$$

that with $s \leqslant \sigma$

$$v(s) = \frac{s-\sigma}{\sigma}.$$

Knowing $v(s)$ with $s \leqslant \sigma$, we find u(s), utilizing communication/connection between these functions

$$u' + v + \frac{v^2}{2} = 0.$$

Taking into account, that u(0)=0, we obtain the following expression for u(s) with $s \leqslant \sigma$:

$$u = -\frac{1}{2\sigma}(s-\sigma)^2 - \frac{1}{6\sigma^2}(s-\sigma)^3 + \frac{\sigma}{3}.$$

For determining the functions u(s) and $v(s)$ with $s>\sigma$ we minimize the functional

$$\int_{\sigma}^{\infty}(v'^2+u^2)\,ds$$

in the nonholonomic constraint between the varied functions

$$u'+v=0.$$

The solution of this task in no way differs from that given in §1, and it gives the following expressions for functions u and v' with $s>\sigma$:

$$u=-\frac{\sigma}{3\sqrt{2}}(\omega_1 e^{\omega_1(s-\sigma)}+\omega_2 e^{\omega_2(s-\sigma)}),$$

$$v'=-\frac{\sigma i}{3\sqrt{2}}(-\omega_1 e^{\omega_1(s-\sigma)}+\omega_2 e^{\omega_2(s-\sigma)}),$$

where

$$\omega_1=\frac{1}{\sqrt{2}}(-1+i),\qquad \omega_2=\frac{1}{\sqrt{2}}(-1-i).$$

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After substituting the obtained values of functions u, v into the expression of functional $\bar{U}$, we will obtain its value depending on the parameter σ

$$\bar{U}(\sigma)=K\left\{\frac{\sqrt{2}\sigma^2}{9}+\left(\frac{1}{20}+\frac{1}{7\cdot 36}\right)\sigma^3\right\}+\frac{\alpha\delta^2\sigma_e}{4}.$$

Further we must minimize expression $\bar{U}(\sigma)$ on σ. However, we see

that $\bar{U}(\sigma)$ monotonically decreases with $\sigma \to 0$. This means that the examine/considered by us task of regular solution does not have, since with $\sigma \to 0$ in cut $(0, \sigma)$

$$v' = \frac{1}{\sigma} \to \infty.$$

The physical sense of the obtained result lies in the fact that the appearance of plastic deformations on the boundary of bulge conducts k against the formation of an actual fin/edge on the surface of shell ($k = 1/\sigma \to \infty$). Not difficult to explain the mechanism of the formation of fin/edge demonstrative reasons. Actually, the elastic state of shell before the appearance of plastic deformations is determined in essence by bending in the plane, perpendicular to fin/edge, and accompanying this bending by the elongation of median surface in outer zone and by compression in internal (Fig. 12). If at point C on fin/edge appear plastic deformations, then the flexural rigidity of shell is decreased. In this case, the elongation of median surface in outer zone and compression in internal increase bending strain in C. This leads to further weakening of shell on bending. As a result the shell takes the form with very large, theoretically infinite curvature in the direction, perpendicular to fin/edge.

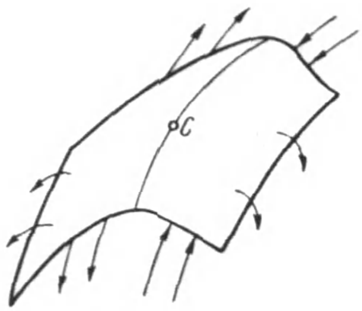

Fig. 12.

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The appearance of plastic deformations on the boundary of bulge stops supercritical deformation. Really/actually, supercritical deformation is accompanied by a change in the region of bulge, and therefore by the displacement/movement of the fin/edge, which limits this region. If at the particular point of fin/edge appear plastic deformations from bending, then curvature k of surface in the direction, perpendicular to fin/edge, becomes very large (in our examination infinite). The displacement/movement of fin/edge is connected with the bending of shell first in one direction to curvature k, and then with the bending in the opposite direction, which virtually reduces initial form. The energy per the unit surface area of shell, connected with this deformation, is equal to

$$\simeq \frac{\sigma_e k \delta^2}{2}.$$

Therefore the displacement of the cell/element of fin/edge Δl to

value Δs requires the execution of the work

$$\frac{\sigma_e k \delta^2}{2} \Delta l \Delta s.$$

This work must fulfill the effective on shell load. But load is final, and value k is very great (is infinitely great). Consequently, $\Delta s=0$, i.e., the appearance of plastic deformations stops supercritical deformation.

Now we can formulate principle A in connection with limited elastic ones shell.

Limitedly elastic shell allow/assumes only such supercritical deformations, determined by principle A with which the voltage/stresses σ on the boundary of the bulges, determined on formula (§1)

$$\sigma = c'E \frac{\delta^{1/2} \alpha^{3/2}}{\rho^{1/2}},$$

do not exceed time/temporary resistances σ_e (we consider that the tensile strength is elastic limit).

Let us use the obtained result for the investigation of the supercritical deformations of strictly convex hulls under external pressure.

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Under the assumptions p. 3, smallest received by shell load p_i during transcritical deformation is determined by the maximum permissible deformation and is calculated from the formula

$$p_i = \frac{3}{2} cE (k_1 + k_2) \sqrt{k_1 k_2} \frac{\delta^{5/2}}{\sqrt{2h_i}}.$$

In the case of unlimitedly elastic shells, maximum deformation $2h_i$ is determined by the geometric dimensions of shell. In the case limitedly elastic shells, maximum deformation can be determined by the condition for appearance on the boundary of the bulge of plastic deformations, i.e., by condition (page 56)

$$c'E \sqrt{2h_i} \delta^{1/2} \sqrt{k_1 k_2} = \sigma_s. \qquad (*)$$

Determining hence $2h_i$ and substituting it in formula for p_i, we will obtain lower critical load for limiting the elastic shells

$$p_i = \frac{3}{2} cc' \left(\frac{E}{\sigma_s}\right) (k_1 + k_2) k_1 k_2 E \delta^3.$$

The application/use of this formula logically assumes that the deformation, determined by condition (*), is geometrically permissible.

In the case of the spherical shell of radius R, the formula for a lower critical load takes the form

$$p_i = 3cc' \left(\frac{E}{\sigma_s}\right) E \left(\frac{\delta}{R}\right)^3.$$

§3 about the stability of the supercritical axially symmetric deformations of spherical shell with axially symmetric loading.

In § 1 in the examination of the supercritical deformations of

the flat strictly convex hulls, rigidly attached on edge, it is shown that such deformations in basic approach/approximation are reduced to mirror bulge. The establishment of this fact substantially rests on two assumptions: 1) the deformation of shell must be considerable and 2) the attachment of the edge of shell is sufficiently rigid. With the disturbance/breakdown at least of one of these conditions, we are right to expect another result. Of this, us convince data of the corresponding experiments.

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The supercritical deformation of spherical shell with the bulge of the region of small size/dimensions, and also with the insufficient rigidity of the attachment of edge can not possess axial symmetry, although the shell and the method of its loading are axially symmetric. The region of bulge frequently has a form of triangle or quadrangle with the rounded off apex/vertexes. In connection with this in present paragraph we want to investigate a question concerning the stability of the axially symmetric supercritical deformations of spherical shell with an axisymmetrical load. Will be examined two load cases of the shell: by uniform external pressure and concentrated force. In the latter case the results of theoretical examination will be compared with the data of the corresponding experiment.

1. Bending of spherical segment. Equations of bending. F - flat spherical segment with single curvature and radius of basis/base $\rho_0 << 1$. Let us introduce the Cartesian coordinates ~~χ~~ x, y, z, after accepting tangential plane in the apex/vertex of segment for plane xy, and internal standard - as the positive semi-axis z. In these coordinates the segment is assigned by the equation

$$x^2 + y^2 + z^2 - 2z = 0.$$

Let us designate through γ the curve on the surface of segment, which for plane ~~XY~~ xy is design/projected into the curve, assigned in polar coordinates r, ϑ with the equation

$$r = \rho(1 + \lambda \cos k\vartheta),$$
$$\rho \ll \rho_0, \quad \lambda \ll 1.$$

Curve γ divide/marks off the surface of segment into two regions: internal - F", limited by curve γ, and external - F'. Let us examine the task of the bending of segment with the extrusion of region F" To the Inside ~~vnutr~~, the formation of the fin/edge of lengthwise curved γ and by the preservation/retention/maintaining of edge in initial plane (Fig. 13). Is proposed the following method of solving the task.

For each of the surfaces F' and F" we will construct the independent bending. Such bending are possible and besides with large arbitrariness.

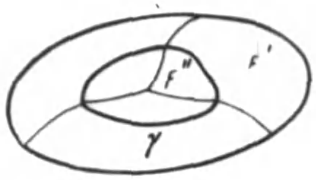

Fig. 13.

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By this arbitrariness we will be ordered so that the curved, limiting ranges F' and F", would prove to be combined. As a result, we will obtain surface with fin/edge lengthwise γ, iscmetric to initial segment. Let us examine the bending of surfaces F' and F".

r - vector of point of one of the surfaces, for example, F', and τ - displacement vector of this point with the bending of surface. Since the linear cell/element of surface with bending does not change, then must occur the equality

$$dr^2=(dr+d\tau)^2.$$

Hence for a vector function τ, is obtained the equation

$$dr\,d\tau+\frac{1}{2}d\tau^2=0.$$

Curve γ on the surface of segment, that converts into fin/edge with bending, depends on the parameter λ. The parameter λ is low, and therefore it is advisable to decompose/expand vector function τ according to the degrees of this parameter

$$\tau=\lambda\tau_1+\lambda^2\tau_2+\ \ldots$$

We set/assume τ_0 equal to zero, since with $\lambda=0$ the task of bending

has trivial solution with the mirror reflection of segment F" relative to plane curved γ. In this field τ on surface F', it is equal to zero identically.

Substituting expansion/decomposition τ according to degrees λ in the equation of bending, we will obtain for vector functions τ_1, τ_2, ... infinite system of equations

$$dr\, d\tau_1 = 0,$$
$$dr\, d\tau_2 + \frac{d\tau_1^2}{2} = 0,$$
$$\cdots\cdots\cdots$$

Vector fields τ_1, τ_2, ... are called the bending fields of the first, second and so forth of orders.

Let us introduce in space the standardized/normalized cylindrical coordinates u, v, z. For the arbitrary point A, they have the following values: $\rho^2 z$ - with an accuracy to the sign of the distance of point A from plane xy. ρu - distance to point A from Z-axis, v - the angle, formed by the plane σ, passing through Z-axis and point A, with plane xz.

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This standardization of coordinates is convenient in the examination of the task of bending in the case of the low region F".

With each point A of space, we will relate the movable trihedron of three single mutually perpendicular vectors e_1, e_2, e_3 (Fig. 14). Vector e_3 is directed along the axis z to side z>0, vector e_1 lie/rests at plane σ and is directed from z-axis, while vector e_2 is perpendicular this plane. It is obvious, vector e_3 does not depend on point A, but vectors e_1, e_2 depend only on coordinate v of this plane. In this case, it is possible to count that vector e_2 is directed so that

$$\frac{de_1}{dv} = e_2.$$

Then, obviously,

$$\frac{de_2}{dv} = -e_1.$$

Let us examine the case of the low regions F", i.e., the case of the low values of the parameter ρ. In this case, it is expedient to standardize the coordinates of displacement vector τ with the aid of the parameter ρ. Specifically,, the components of vector τ relative to basis e_1, e_2, e_3 are conveniently presented in the following form:

$$\rho^3\xi, \quad \rho^3\eta, \quad \rho^2\zeta.$$

Equation of the bending

$$dr\,d\tau + \frac{d\tau^2}{2} = 0$$

of the equivalently to the system three equations

$$\left.\begin{aligned} r_u\tau_u + \frac{\tau_u^2}{2} &= 0, \\ r_u\tau_v + r_v\tau_u + \tau_u\tau_v &= 0, \\ r_v\tau_v + \frac{\tau_v^2}{2} &= 0. \end{aligned}\right\} \qquad (*)$$

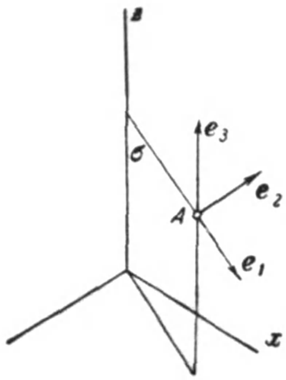

Fig. 14.

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If we into the first equation of this system substitute vectors r and τ, decomposed in basis e_1, e_2, e_3, precisely,

$$r = \rho u e_1 + \rho^2 z e_3,$$
$$\tau = \rho^3 \xi e_1 + \rho^3 \eta e_2 + \rho^2 \zeta e_3,$$

and we will use the formulae of differentiation for the vectors of the basis

$$\frac{de_1}{dv} = e_2, \quad \frac{de_2}{dv} = -e_1,$$

then we will obtain

$$\xi_u + z_u \zeta_u + \frac{1}{2}\zeta_u^2 + \frac{\rho^2}{2}(\xi_u^2 + \eta_u^2) = 0.$$

With small ρ last/latter term/component/addend in this equation can be reject/thrown. Furthermore, from the equation of the segment

$$\rho^2 z = 1 - \sqrt{1 - \rho^2 u^2}$$

with small ρ is obtained

$$z_u \simeq u.$$

As a result the equation for ξ, η, ζ at low values ρ can be presented in the form

$$\xi_u + u\zeta_u + \frac{1}{2}\zeta_u^2 = 0.$$

It is analogous, of other two equations of bending after the substitution of values r and, τ at the low values of the parameter ρ is obtained

$$u(\eta_v + \xi) + \frac{1}{2}\zeta_v^2 = 0,$$
$$\xi_v - \eta + u\zeta_v + u\eta_u + \zeta_u\zeta_v = 0.$$

In the examination of the bending of surface F' us must solve system of equations (*) for functions ξ, η, ζ in the region

$$1 + \lambda \cos kv \leqslant u \leqslant \frac{\rho_0}{\rho}$$

under the boundary condition

$$\zeta\left(\frac{\rho_0}{\rho}\right) = 0.$$

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However, in view of the predicted smallness of the parameter ρ, we will search for solution in the region, determined only by first inequality, and boundary condition let us relate to infinity, i.e., let us consider that $\zeta(u) \to 0$ with $u \to \infty$.

Analogous examinations can be conducted for the bending of another part of the segment - F". In this case, is obtained in accuracy/precision the same system of equations for functions ξ, η, ζ. But its solution must be examined on the remaining part of the plane, i.e., in the region

$$u \leqslant 1 + \lambda \cos kv.$$

In order for the surfaces obtained by bending regions F' and F", was comprised the surface, the isometric segment with fin/edge lengthwise γ, is necessary that the corresponding displacement r' and r'' in the regions indicated on their overall boundary

$$u = 1 + \lambda \cos kv$$

would satisfy the conditions

$$\xi' = \xi'', \quad \eta' = \eta'', \quad \zeta' + \zeta'' + u^2 = 0.$$

During the execution of these conditions, the edge of the surface $\tilde{F}''$, obtained by bending from F", after mirror reflection in plane xy will be combined with edge of surface $\tilde{F}'$, obtained by bending from F', and is formed the interesting us the surface with fin/edge, isometric to segment.

Subsequently conditions indicated above for solutions of r' and r'' for curve u=1+λ cos kv will be called the conditions of coupling.

2. Solution of equations of bending. The common/general/total plan/layout of the solution of the task of the bending of segment will consist of following. First of all, we note that $\zeta_0'' = -1$. This directly escape/ensues from the condition of the coupling

$$\zeta' + \zeta'' + u^2 = 0.$$

As far as components are concerned two others ξ_0'' and η_0'', without

limiting generality, them it is possible to take as equal to zero. This can always be achieved by motion, parallel to plane xy.

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Further, we will determine the bending first-order fields r'_1, r''_1 for surfaces F', F", and by the arbitrariness which in this case is obtained, we will be ordered that so that the conditions of coupling would be satisfied with an accuracy to the values of order λ. Then we determine the bending fields of the second order, satisfying the conditions of coupling with an accuracy to λ^2, and so forth.

The bending fields of the first order satisfy the system of equations

$$\xi_u + u\zeta_u = 0,$$
$$\eta_v + \xi = 0,$$
$$\xi_v - \eta + u\zeta_v + u\eta_u = 0.$$

If we from these three equations exclude functions ξ and η, then for ζ is obtained the equation

$$\frac{\zeta_{vv}}{u} + (u\zeta_u)_u = 0.$$

It is the equation of Laplace in polar coordinates u, v.

Let us search for the solution of equation for ζ in the form of

trigonometric series. In this case, in view of the symmetry of the expected solution, it is possible to count that in this expansion/decomposition are present only the terms, which contain the cosines of arcs, multiple kv. Thus,

$$\zeta=\sum c_n(u)\cos nv,$$

where n accepts only integral multiple k of value. The general solution for ζ in this form takes the form

$$\zeta=\sum\left(\frac{a_n}{u^n}+b_n u^n\right)\cos nv.$$

In the case of the bending of surface F', it is necessary to assume $a_0=0$ and $b_n=0$, since $\zeta\to 0$ with $u\to\infty$. Thus, for component ζ of the bending field r' of surface F' is obtained the expression

$$\zeta=\sum\frac{a_n}{u^n}\cos nv,$$

where the addition begins with n>0 and goes over by the whole n, by multiple k.

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According to the considerations of convenience, we will give ζ the following form:

$$\zeta=\sum(-a_n)\frac{n-1}{n}\frac{\cos nv}{u^n}.$$

Having ζ, it is not difficult to find from the equations of bending two other components ξ and η. It is obtained

$$\xi=\sum a_n\frac{\cos nv}{u^{n-1}},\quad \eta=\sum(-a_n)\frac{\sin nv}{nu^{n-1}}.$$

Integration constant are accepted equal to zero due to the predicted

symmetry of bending.

In the case of the bending field τ_1'' surface F" in the expression

$$\zeta=\sum\left(\frac{a_n}{u^n}+b_n u^n\right)\cos nv$$

it is necessary to place equal to zero constants a_n, since ζ must be limited in zero (with u=0). The absolute term of expansion b_0 also it is possible to consider it equal to zero, since it corresponds to the simple shift of surface as whole, and this shift is taken into account especially. Just as for surface F', according to the considerations of convenience, ζ for surface of F" we represent in the form

$$\zeta=\sum(-b_n)\frac{n+1}{n}u^n\cos nv.$$

With the aid of the system of equations of bending and obtained expression for ζ, we find ξ and η

$$\xi=\sum b_n u^{n+1}\cos nv,\quad \eta=\sum(-b_n)\frac{u^{n+1}}{n}\sin nv.$$

Let us find now the solutions, which satisfy the conditions of coupling with an accuracy to the values of order λ. The conditions of coupling can be written in the form

$$\lambda\xi_1'+O(\lambda^2)=\lambda\xi_1''+O(\lambda^2),$$
$$\lambda\eta_1'+O(\lambda^2)=\lambda\eta_1''+O(\lambda^2),$$
$$\lambda\zeta_1'+\lambda\zeta_1''+\zeta_0''+1+2\lambda\cos kv+O(\lambda^2)=0.$$

$(\zeta''_0=-1)$.

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Isolating in these equalities the terms, which are of the order λ, we will obtain

$$\sum a_n \cos nv = \sum b_n \cos nv,$$
$$\sum \left(-\frac{a_n}{n}\right) \sin nv = \sum \left(-\frac{b_n}{n}\right) \sin nv,$$
$$\sum (-a_n) \frac{n-1}{n} \cos nv + \sum \left(-b_n\right) \frac{n+1}{n} \cos nv + 2\cos kv = 0.$$

From the first two equalities it follows that

$$a_n = b_n.$$

From the third equation is obtained with n>k

$$a_n \frac{n-1}{n} + b_n \frac{n+1}{n} = 0,$$

and at n=k

$$-a_k \frac{k-1}{k} - b_k \frac{k+1}{k} + 2 = 0.$$

We hence consist that with n>k all a_n and b_n are equal to zero, but with n=k they are determined from the system

$$a_k = b_k, \quad -a_k \frac{k-1}{k} - b_k \frac{k+1}{k} + 2 = 0.$$

Solution of this system following:

$$a_k = b_k = 1.$$

In such a way as to satisfy the condition of coupling with an accuracy to the values of order λ, it is necessary to take the bending first-order fields in the following form.

For surface F':

$$\xi_1 = \frac{\cos kv}{u^{k-1}}, \quad \eta_1 = -\frac{\sin kv}{ku^{k-1}}, \quad \zeta_1 = -\frac{k-1}{k}\frac{\cos kv}{u^k},$$

For surface of F":

$$\xi_1 = u^{k+1}\cos kv, \quad \eta_1 = -\frac{u^{k+1}}{k}\sin kv, \quad \zeta_1 = -\frac{k+1}{k}u^k\cos kv.$$

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Let us turn now to the bending fields of second order. They satisfy the system

$$\xi_u + u\zeta_u + \frac{1}{2}\zeta_{1u}^2 = 0,$$
$$u(\eta_v + \xi) + \frac{1}{2}\zeta_{1v}^2 = 0,$$
$$\xi_v - \eta + u\zeta_v + u\eta_u + \zeta_{1u}\zeta_{1v} = 0.$$

This system is heterogeneous relative to the unknown functions, and its general solution is obtained by the addition of any of particular solution and general solution of the corresponding uniform system. The latter is system of equations for the which bends fields first order.

By the additions of the bending fields of first order when obtaining general solutions for the bending fields of the second order we will be ordered so as to satisfy the conditions of coupling with an accuracy to the values of order λ^2. It proves to be, these

additions by the condition indicated are determined unambiguously.

Lowering the appropriate unpacking/facings, let us give expressions for the components of displacement τ with an accuracy to second-order quantities in the parameter λ.

For surface F':

$$\xi = \frac{\lambda \cos kv}{u^{k-1}} - \lambda^2 \frac{(k-1)^2}{4u^{2k+1}} + \ldots,$$
$$\eta = -\frac{\lambda \sin kv}{ku^{k-1}} - \ldots,$$
$$\zeta = -\lambda \frac{k-1}{k} \frac{\cos kv}{u^k} + \lambda^2 \frac{(k-1)^2}{4u^{2k+2}} + \ldots$$

For surface of F":

$$\xi = \lambda u^{k+1} \cos kv - \frac{\lambda^2}{4}(k+1)^2 u^{2k-1} + \ldots,$$
$$\eta = -\frac{\lambda u^{k+1}}{k} \sin kv + \ldots,$$
$$\zeta = -1 - \lambda \frac{k+1}{k} u^k \cos kv + \frac{\lambda^2}{4}(k+1)^2 u^{2k-2} - \frac{\lambda^2 k^2}{2} + \ldots$$

Here are not everywhere extracted the members of the form

$$\lambda^2 A \cos 2kv, \quad \lambda^2 B \sin 2kv.$$

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They although are of the order λ^2, in our further examinations they are unessential.

3. Determination of some values for surface, obtained by bending

of spherical segment. In p. 2, we found expressions for the components of the displacement vector τ of the point of spherical segment with its bending. Their values are obtained with an accuracy to the values of order λ^2. Us subsequently, they will interest some integral expressions of the form

$$J = \int\int \varphi(\tau, \tau_u, \tau_v \ldots)\, du\, dv,$$

where the integration for v is fulfilled within the limits of 0.2π. It is obvious, if we take τ with an accuracy to the values of order λ^2, then with the same accuracy/precision we will obtain value J.

It proves to be, if we in expression τ drop/omit the members of the form

$$\lambda^2 \cos 2kv, \quad \lambda^2 \sin 2kv,$$

then with an accuracy to the values of order λ^2 is obtained the same value J. Really/actually, connected with this process/operation change in the integrand will take the form

$$\lambda^2 C' \cos 2kv + \lambda^2 C'' \sin 2kv + O(\lambda^3),$$

where C' and C" they do not depend on v. But during the integration of this expression is obtained the value of order λ^3. For this very reason at the end of the preceding/previous point/item we gave the simplified expression for functions ξ, η, ζ, after drop/omitting in them the members of order λ^2, having the form

$$\lambda^2 A \cos 2kv, \quad \lambda^2 B \sin 2kv.$$

Let us designate through $\tilde{F}'$ and $\tilde{F}''$ the parts of the isometric to the segment of the surfaces, which correspond on isometry to regions F' and F". Let us find the equations of these surfaces. So as the standardized/normalized components of the shift of point with the bending of segment are equal to ξ, η, ζ, that true shift will be

$$\rho^3\xi, \quad \rho^3\eta, \quad \rho^2\zeta.$$

Therefore the equation of surface $\tilde{F}'$ can be written thus:

$$r = e_1(\rho u + \rho^3\xi') + e_2(\rho^3\eta') + e_3\left(\frac{\rho^2u^2}{2} + \rho^2\zeta'\right),$$
$$u \geqslant 1 + \lambda\cos kv.$$

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Respectively the equation of surface $\tilde{F}''$ will be

$$r = e_1(\rho u + \rho^3\xi'') + e_2(\rho^3\eta'') - e_3\left(\frac{\rho^2u^2}{2} + \rho^2\zeta''\right),$$
$$u \leqslant 1 + \lambda\cos kv.$$

In view of the fact that we assume the parameter ρ sufficient to small ones, the members of order ρ^3 in the equations of surfaces it is possible to drop/omit. Then we obtain: for a surface $\tilde{F}'$

$$r = e_1(\rho u) + e_3\left(\frac{\rho^2u^2}{2} + \rho^2\zeta'\right),$$

for a surface $\tilde{F}''$

$$r = e_1(\rho u) - e_3\left(\frac{\rho^2u^2}{2} + \rho^2\zeta''\right).$$

Surfaces $\tilde{F}'$ and $\tilde{F}''$, forming isometric to segment surface, are divided by fin/edge γ. Let us determine angle α with this fin/edge between F' and F". For this, let us first find the unit vectors of standards n' and n" surfaces along fin/edge.

We have

$$n' = \frac{r'_u \times r'_v}{|r'_u \times r'_v|}, \quad n'' = \frac{r''_u \times r''_v}{|r''_u \times r''_v|}.$$

In view of the fact that the surface $\bar{F}$ is obtained by the isometric conversion of segment F, the denominators of these formulas at the points of fin/edge γ have the same value, as for a segment. A for it with small ones ρ

$$|r_u \times r_v| \simeq \rho^2 u.$$

Let us calculate derivatives of a vector r' according to u and v. We have

$$r'_u = e_1\rho + e_3\rho^2 (u + \zeta_u),$$
$$r'_v = e_2\rho u + e_3\rho^2\zeta_v.$$

Hence, determining vector multiplication by the equalities

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2,$$

we will obtain

$$r'_u \times r'_v = e_3 (\rho^2 u) - e_2 (\rho^3\zeta'_v) - e_1 (\rho^3 u^2 + \rho^3 u\zeta'_u).$$

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Analogously we find

$$r''_u \times r''_v = e_3 (\rho^2 u) + e_2 (\rho^3\zeta''_v) + e_1 (\rho^3 u^2 + \rho^3 u\zeta''_u).$$

Thus,

$$n' = e_3 - e_2 \left(\frac{\rho}{u}\zeta'_v\right) - e_1\rho (u + \zeta'_u).$$

$$n'' = e_3 + e_2 \left(\frac{\rho}{u}\zeta''_v\right) + e_1\rho (u + \zeta''_u).$$

But now, taking into account, that

$$(n' \times n'')^2 = \sin^2 \alpha,$$

where α - angle with fin/edge γ, we find

$$\sin^2 \alpha = \rho^2 \left\{ \left(\frac{\zeta_v' + \zeta_v''}{u} \right)^2 + (2u + \zeta_u' + \zeta_u'')^2 \right\} + \ldots,$$

where are not extracted the terms, which are of the order ρ^4. In view of smallness ρ with the same accuracy/precision (ρ^4) we have

$$\alpha^2 = \rho^2 \left\{ \left(\frac{\zeta_v' + \zeta_v''}{u} \right)^2 + (2u + \zeta_u' + \zeta_a'')^2 \right\} + \ldots$$

Let us find the curvature of fin/edge γ on surface $\tilde{F}$. It it is possible to express by geodetic curvature k_g and angle with fin/edge. In view of the isometry of surfaces F and $\tilde{F}$ geodetic curvature k_g can be calculated on initial surface. In this case, it is obvious that with small ρ the geodetic curvature k_g, with an accuracy to the values of higher order relatively ρ, is equal to the usual curvature of curved not plane, assign/prescribed in polar coordinates by the equation

$$r = \rho (1 + \lambda \cos kv).$$

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The latter is calculated from known formula and for it is obtained the following expression:

$$k_g = \frac{1}{\rho} \frac{\left| 1 + \lambda (2 + k^2) \cos kv + \frac{\lambda^2}{2} (1 + 3k^2) \right|}{\left(1 + 2\lambda \cos kv + \frac{\lambda^2}{2} (1 + k^2) \right)^{3/2}} + \ldots$$

Here are not extracted the members of order above λ^2 and the members of the order λ^2 of the form

$$\lambda^2 A \cos 2kv, \quad \lambda^2 B \sin 2kv.$$

In view of smallness ρ, $\alpha \sim \rho$, and it is possible to count that for curvature k of fin/edge on surface of $\tilde{F}$ we have

$$k \simeq k_g$$

This equality transfer/converts into precise with $\rho \to 0$.

Let us calculate the cell/element of arc ds along the fin/edge γ of surface $\tilde{F}$. Taking into account the isometry of surfaces $\tilde{F}$ and F, with small ones ρ the cell/element of arc ds along fin/edge γ can be calculated according to the formula

$$ds = ds_e + \dots,$$

where ds_e - a cell/element of the arc of curve, assign/prescribed in polar coordinates by the equation

$$r = \rho(1 + \lambda \cos kv),$$

and the nonextracted terms have higher order of smallness on ρ. Thus,

$$ds = \rho\left[(1 + 2\lambda \cos kv) + \frac{\lambda^2}{2}(1 + k^2)\right]^{1/2} dv.$$

Subsequently by us will be necessary the mean curvature of

surface $\tilde{F}$. Let us calculate it. On known formula the mean curvature of surface is equal to

$$H=\frac{1}{2}\frac{LG-2FM+NE}{EG-F^2},$$

where to E, F, G, L, M, N - coefficients of the first and second quadratic shapes of surface. As far as coefficients are concerned of the first quadratic form, they the same as on initial surface (spherical segment).

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Consequently, with small ρ

$$E=\rho^2,\quad F=0,\quad G=\rho^2u^2.$$

Let us find coefficients of L and N of the second quadratic form. For a surface $\tilde{F}'$

$$\boldsymbol{r}_{uu}=\boldsymbol{e}_3(\rho^2+\rho^2\zeta_{uu}),$$

$$\boldsymbol{n}=\boldsymbol{e}_3-\boldsymbol{e}_2\left(\frac{\rho}{u}\zeta_v\right)-\boldsymbol{e}_1(\rho u+\rho\zeta_u).$$

Hence

$$L=\boldsymbol{r}_{uu}\boldsymbol{n}=\rho^2+\rho^2\zeta_{uu}.$$

Analogously it is obtained

$$N=\boldsymbol{r}_{vv}\boldsymbol{n}=\rho^2\zeta_{vv}+\rho^2u(u+\zeta_u).$$

For a surface $\tilde{F}''$ are obtained the same expressions of coefficients L and N, only with their function ζ and opposite sign.

Substituting the obtained expressions of the coefficients of quadratic shapes of surface in formula for a mean curvature, we find:

for a surface $\tilde{F}'$

$$H=\frac{1}{2u^2}\left\{u^2(1+\zeta'_{uu})+\zeta'_{vv}+u(u+\zeta'_u)\right\}.$$

for a surface $\tilde{F}''$

$$H=-\frac{1}{2u^2}\left\{u^2(1+\zeta''_{uu})+\zeta''_{vv}+u(u+\zeta''_u)\right\}.$$

In conclusion let us note also that the Gaussian surface curvature $\tilde{F}$ is equal to the Gaussian curvature of initial surface, therefore, it is constant and equal to unity.

4. Strain energy of shell. For the strain energy of shell in §1 we obtained the following formula (page 33):

$$U=\int_{\gamma} cE\delta^{5/2}\alpha^{5/2}k^{1/2}\,ds_{\gamma}+\frac{E\delta^3}{6(1-\nu^2)}\int_{\gamma}\alpha\left(-k_n+\frac{k_l+k_e}{2}\right)ds_{\gamma}+$$
$$+\frac{E\delta^3}{24(1-\nu^2)}\int_{\tilde{F}}\int(\Delta k_1^2+\Delta k_2^2+2\nu\,\Delta k_1\,\Delta k_2)\,d\sigma.$$

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Let us designate for brevity term/component/addends of the right side of this formula through U', U'' and U''' respectively, and let us calculate consecutively each of these term/component/addends.

The values, entering the formula for U', have calculated we in p. 2. It is necessary, however, to keep in mind that the obtained there values are related to the sphere of a single radius. Therefore for the shell in question the radius of curvature of which we will designate R, linear values must be increased in R times.

In p. 3 for an angle α_{γ}, formed by the tangential planes of surface $\tilde{F}$ along fin/edge γ, obtained following expression

$$\alpha_{\gamma}^2=\rho^2\left\{\left(\frac{\zeta_v'+\zeta_v''}{u}\right)^2+(2u+\zeta_u'+\zeta_u'')^2\right\}.$$

Taking into account, that an angle α, entering term/component/addend U', two times less α_{γ}, let us have

$$\alpha^2=\rho^2\left\{\left(\frac{\zeta_v'+\zeta_v''}{2u}\right)^2+\left(u+\frac{\zeta_u'+\zeta_u''}{2}\right)^2\right\}.$$

Substituting here

$$u = 1 + \lambda \cos kv,$$

$$\zeta' = -\frac{\lambda(k-1)}{k}\frac{\cos kv}{u^k} + \frac{\lambda^2(k-1)^2}{4u^{2k+2}} + \dots$$

$$\zeta'' = -1 - \frac{\lambda(k+1)}{k}u^k \cos kv + \frac{\lambda^2(k+1)^2}{4}u^{2k-2} - \frac{\lambda^2 k^2}{2} + \dots,$$

let us have

$$\alpha^2 = \rho^2\left(1 + \frac{\lambda^2 k^2}{2} + \dots\right).$$

Here, as everywhere subsequently in the analogous cases, are not extracted the members of order above λ^2 and the members of the order λ^2 of the form

$$\lambda^2 A \cos 2kv, \quad \lambda^2 B \sin 2kv.$$

Let us calculate now

$$\sqrt{k}\,ds.$$

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Taking into account the obtained in p. 2 expressions for k and ds, let us find

$$\sqrt{k}\,ds = \sqrt{\rho}\,\frac{\left|1 + \lambda(k^2+2)\cos kv + \frac{\lambda^2}{2}(3k^2+1)\right|^{1/2}}{\left(1 + 2\lambda\cos kv + \frac{\lambda^2}{2}(k^2+1)\right)^{1/4}}\,dv.$$

If we the right side of the equality decompose according to degrees λ, after drop/omitting the members of order above λ^2 and the unessential members of order λ^2, then we will obtain

$$\sqrt{k}\,ds = \sqrt{\rho}\left\{1 + \frac{\lambda}{2}(k^2+1)\cos kv + \lambda^2\left(\frac{3k^2}{8} - \frac{1+k^2}{16}\right)\right\}dv.$$

For the sphere of radius R result must be multiplied on $\sqrt{R}$. Thus, for

the shell in question we have

$$\sqrt{k}\,ds = \sqrt{\rho R}\left\{1 + \frac{\lambda}{2}(k^2+1)\cos kv + \lambda^2\left(\frac{3k^2}{8} - \frac{1+k^2}{16}\right)\right\} dv.$$

Substituting the value α and $\sqrt{k}\,ds$ in formula for U', we will obtain

$$U' = 2\pi c E \delta^{5/2} R^{1/2} \rho^3 \left\{1 + \lambda^2\left(k^2 - \frac{(1+k^2)^2}{16}\right)\right\}.$$

Let us study now expression U''. In connection with this let us, first of all, find normal curvatures k_i and k_e.

As is known, normal curvatures are calculated from the formula

$$k = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2},$$

where E, F, G, L, M, N - coefficients of the first and second quadratic shapes of surface. For the examine/considered by us surface of $\tilde{F}$, obtained by the bending of spherical segment, we have

$$E = \rho^2, \quad F = 0, \quad G = \rho^2 u^2.$$

As far as coefficients are concerned of the second quadratic form, for surface $\tilde{F}'$ will be

$$L' = \rho^2(1 + \zeta'_{uu}),$$
$$M' = \rho^2\left(\zeta'_{uv} - \frac{\zeta'_v}{u}\right),$$
$$N' = \rho^2(\zeta'_{vv} + u^2 + u\zeta'_u).$$

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But for surface of $\tilde{F}''$ we have

$$L'' = -\rho^2 (1 + \zeta''_{uu}),$$
$$M'' = -\rho^2 \left(\zeta''_{uv} - \frac{\zeta''_v}{u}\right),$$
$$N'' = -\rho^2 (\zeta''_{vv} + u^2 + u\zeta''_u).$$

Since curved γ in coordinates u, V it is assigned by the equation

$$u = 1 + \lambda \cos kv,$$

its the direction at the arbitrary point u, V will be

$$du : dv = -\lambda k \sin kv : 1.$$

Consequently, perpendicular direction, that is the direction, in which are measured normal curvatures k_i and k_e, will be

$$du : dv = u^2 : (\lambda k \sin kv).$$

Substituting the obtained values of the coefficients of quadratic forms and du, dv into formula for normal curvatures k_i, k_e and by holding only essential terms, we will obtain

$$k_e = 1 + \zeta'_{uu} + 2(\zeta'_{uv} - \zeta'_v)\lambda k \sin kv + \ldots$$
$$k_i = -1 - \zeta''_{uv} - 2(\zeta''_{uv} - \zeta''_v)\lambda k \sin kv + \ldots$$

Hence

$$k_i + k_e = (\zeta'_{uu} - \zeta''_{uu}) + 2\lambda k (\zeta'_{uv} - \zeta''_{uv}) \sin kv - \\ - 2\lambda k (\zeta'_v - \zeta''_v) \sin kv + \ldots$$

All this is related to the segment of single curvature; for the shell in question the obtained value $k_i + k_e$ must be divided into radius R.

The cell/element of arc to curve γ is equal to

$$ds_\gamma = R\rho\,(\lambda^2 k^2 \sin^2 kv + u^2)^{1/2}\,dv = \\ = R\rho\left\{u\left(1 + \frac{\lambda^2 k^2}{4} + \ldots\right\}dv.$$

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Taking into account the obtained expressions for α, $k_i + k_e$ and ds_γ, we find

$$\int_\gamma \alpha\,(k_i + k_e)\,ds_\gamma = 0.$$

Thus,

$$U'' = -\frac{E\delta^3}{6\,(1-\nu^2)}\,\frac{1}{R}\int_\gamma \alpha\,ds_\gamma.$$

Substituting here expressions α and ds_γ, we will obtain

$$U'' = -\frac{2\pi E\delta^3\rho^2}{6\,(1-\nu^2)}\left(1 + \frac{\lambda^2 k^2}{2}\right).$$

Let us calculate now the expression

$$U''' = \frac{E\delta^3}{24\,(1-\nu^2)}\int\int_{\tilde{F}} \left(\Delta k_1^2 + \Delta k_2^2 + 2\nu\,\Delta k_1\,\Delta k_2\right)d\sigma.$$

In order to simplify unpacking/facings, let us count Poisson ratio $\nu = 0$. We have

$$\Delta k_1 = \frac{1}{R} - k_1, \quad \Delta k_2 = \frac{1}{R} - k_2,$$

where k_1 and k_2 - principal curvatures of surface $\tilde{F}$. Hence per the unit surface area, we will obtain

$$\bar{U}''' = \frac{E\delta^3}{24}\left(\frac{2}{R^2} - 2K - \frac{4H}{R} + 4H^2\right),$$

where K - Gaussian, and H - mean curvature of surface $\tilde{F}$. Since with the bending of surface its Gaussian curvature does not change, then

$$K = \frac{1}{R^2}.$$

Consequently,

$$\overline{U}''' = \frac{E\delta^3}{6}\left(H^2 - \frac{H}{R}\right).$$

In p. 3 for mean curvature H of surface $\widetilde{F}'$, we found the expression

$$H = 1 + \frac{\zeta'_{uu}}{2} + \frac{\zeta'_{vv}}{2u^2} + \frac{\zeta'_u}{2u}.$$

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Since our shell has radius R, then it with similar bending will have the mean curvature

$$H = \frac{1}{R}\left(1 + \frac{\zeta'_{uu}}{2} + \frac{\zeta'_{vv}}{2u^2} + \frac{\zeta'_u}{2u}\right).$$

The mean curvature of surface $\widetilde{F}''$ is calculated from the same formula, but with its function ζ and with opposite sign.

Let us assume

$$U_1''' = \frac{E\delta^3}{6}\int_{\widetilde{F}'}\int\left(H^2 - \frac{H}{R}\right)d\sigma,$$

$$U_2''' = \frac{E\delta^3}{6}\int_{\widetilde{F}''}\int\left(H^2 - \frac{H}{R}\right)d\sigma.$$

Since the element of area of single sphere is equal to $\rho^2 u\, du\, dv$, then for the shell in question element of area will be

$$d\sigma = R^2\rho^2 u\, du\, dv,$$

therefore

$$U_1''' = \frac{E\delta^3}{6}\int\int\left(H^2 - \frac{H}{R}\right)R^2\rho^2 u\, du\, dv,$$

where the integration is fulfilled on the region of the variables u,

v, determined by the inequality

$$u > 1 + \lambda \cos kv.$$

For convenience in the forthcoming unpacking/facings, we convert integrand in U_1''' as follows:

$$H^2 - \frac{H}{R} = \left(H - \frac{1}{R}\right)^2 + \frac{1}{R}\left(H - \frac{1}{R}\right).$$

For $\widetilde{F'}$ we have

$$\zeta = -\frac{\lambda (k-1)}{k} \frac{\cos kv}{u^k} + \frac{\lambda^2 (k-1)^2}{4u^{2k+2}}.$$

Consequently,

$$H = \frac{1}{R}\left(1 + \frac{\lambda^2 (k-1)^2}{2u^{2k+4}} + \ldots\right).$$

Let us assume

$$u(v) = 1 + \lambda \cos kv.$$

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Then

$$\int\limits_{u(v)}^{\infty} \left(\frac{H}{R} - \frac{1}{R^2}\right) R^2 \rho^2 u \, du = \frac{\lambda^2 \rho^2}{4} (k-1)^2 (k+1) + \ldots,$$

$$\int\limits_{u(v)}^{\infty} \left(H - \frac{1}{R}\right)^2 R^2 \rho^2 u \, du = 0 + \ldots$$

Hence

$$\int\int\limits_{\widetilde{F'}} \left(\frac{H}{R} - \frac{1}{R^2}\right) d\sigma = \frac{\pi \lambda^2 \rho^2}{2} (k-1)^2 (k+1) + \ldots$$

$$\int\int\limits_{\widetilde{F'}} \left(H - \frac{1}{R}\right)^2 d\sigma = 0 + \ldots$$

Consequently, with an accuracy to the values of order λ^2 will be

$$U_1''' = \frac{E\delta^3}{12}\pi\lambda^2\rho^2(k-1)^2(k+1).$$

Value U_2''' is calculated analogously. For is not obtained the following expression:

$$U_2''' = \frac{\pi\rho^2 E\delta^3}{6}\left\{2+\lambda^2\left[1+\frac{3}{2}(k+1)^2(k-1)\right]\right\}.$$

Thus,

$$U''' = \frac{\pi\rho^2 E\delta^3}{6}\{2+\lambda^2(2k^3-2k+k^2)\}.$$

Summarizing term/component/addends U', U" and U''', we find the following expression for strain energy:

$$U = 2\pi c E\delta^{5/2}R^{1/2}\rho^3\left\{1+\lambda^2\left(k^2-\frac{(1+k^2)^2}{16}\right)\right\}+ + \frac{\pi\rho^2 E\delta^3\lambda^2}{3}(k^3-k).$$

5. Work of external load during deformation of shell. We will examine two load cases of the shell: 1) loading by concentrated force f and 2) uniform loading by the external pressure p.

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By loading by concentrated force f, which effects on the internal standard of segment in its center, the produced by it work is the wound

$$A = fh.$$

where h - sagging/deflection in the center of bulge. Taking into account the equation of surface of $\tilde{F}$, obtained in p. 3, we see that

$$h = R\rho^2\left(1+\frac{\lambda^2 k^2}{2}\right).$$

Consequently,

$$A = fR\rho^2\left(1 + \frac{\lambda^2 k^2}{2}\right).$$

In the case of the uniform external pressure p on shell for the produced by it work A we have

$$A = pV,$$

where V - a change (during deformation) in the volume, limited by shell.

Volume change is equal

$$V = \iint_{\tilde{F}} \Delta z \, d\sigma,$$

where Δz - sagging/deflection of shell in the direction of Z-axis during its deformation into form of $\tilde{F}$. on surface $\tilde{F}'$

$$\Delta z = \rho^2 \zeta'.$$

On surface $\tilde{F}''$

$$\Delta z = -(u^2\rho^2 + \rho^2\zeta'').$$

For computing value V, is convenient to break it into two parts of V' and V'', with respect to the separation of range of integration by the curve $u = 1 + \lambda \cos kv$. Let us calculate value V'. We have

$$V' = \int_0^{2\pi} \int_{u(v)}^{\infty} \rho^4 \zeta' u \, du \, dv.$$

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Substituting here

$$\zeta' = -\frac{\lambda(k-1)}{k}\frac{\cos kv}{u^k} + \frac{\lambda^2(k-1)^2}{4u^{2k+2}} + \ldots,$$

we will obtain with an accuracy to the values of order λ^2 the following expression for V':

$$V' = \pi\lambda^2\rho^4\left(\frac{k-1}{k} + \frac{(k-1)^2}{4k}\right).$$

Let us calculate volume V". We have

$$V'' = \int_0^{2\pi}\int_0^{u(v)} -\rho^4(u^2+\zeta'')\,u\,du\,dv.$$

Substituting here

$$\zeta'' = -1 - \frac{\lambda(k+1)}{k}u^k\cos kv + \frac{\lambda^2(k+1)^2}{4}u^{2k-2} - \frac{\lambda^2k^2}{2} + \ldots,$$

we will obtain

$$V'' = \frac{\pi\rho^4}{2} + \pi\lambda^2\rho^4\left\{-1 + \frac{k+1}{k} - \frac{(k+1)^2}{4k} + \frac{k^2}{2}\right\}.$$

Store/adding up V' and V", we find V:

$$V = \frac{\pi\rho^4}{2}(1+\lambda^2k^2).$$

For the shell of radius R this result must be multiplied by R^3. Thus, a change in the limited by shell volume is equal

$$V = \frac{\pi\rho^4R^3}{2}(1+\lambda^2k^2).$$

Consequently, the produced by the external pressure p work is equal to

$$A = \frac{\pi\rho^4R^3}{2}(1+\lambda^2k^2)\,p.$$

6. On stability of axially symmetric deformations of spherical shell. Experiment shows that the spherical shell under the action of concentrated force experience/tests axially symmetric deformation to

the specific torque/moment. When the acting force reaches certain critical value, the axial symmetry of deformation gradually is lost. The region of bulge instead of the circle takes first the form of triangle with the smoothed apex/vertexes, then quadrangle, etc.

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Analogously is matter, also, in the case of uniform loading. The considerable deformation, not troubled by the nearness of edge, differs from axially symmetric and has star structure. We investigate the conditions for transition to the deformations, which do not possess axial symmetry, after accepting as initial ones axially symmetric deformations.

Critical axially symmetric deformation is characterized by the presence of the close forms of equilibrium, possessing axial symmetry. We will search for these forms among the isometric conversions, constructed in p. 2.

According to principle A, the supercritical state of equilibrium under this load is determined from stability condition of the functional

$$W = U - A$$

on many isometric conversions of initial form. In the case

in question

$$U = 2\pi c E\delta^{1/2} R^{1/2} \rho^3 \left\{ 1 + \lambda^2 \left(k^2 - \frac{(1+k^2)^2}{16} \right) \right\} + \\ + \frac{\pi\rho^2 E\delta^3\lambda^2}{3} (k^3 - k).$$

With the loading of shell by concentrated force f in the center of segment the work is equal to

$$A = fR\rho^2 \left(1 + \frac{\lambda^2 k^2}{2} \right).$$

For a shell, poised, the parameters ρ and λ, which characterize deformation, are determined from system of equations

$$\frac{\partial}{\partial\rho}(U - A) = 0, \quad \frac{\partial}{\partial\lambda}(U - A) = 0.$$

With that fix/recorded ρ this system relative to f and λ always has solution with $\lambda=0$ (axially symmetric deformation). If ρ is sufficiently small, then this solution will be only. This means that during small deformation the region of bulge has a form of circle. On the contrary, during large deformations (that is with large ρ) system admits solution with $\lambda \neq 0$. The value ρ, which demarcates these two cases, determines critical axially symmetric deformation.

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Thus, it is determined by the conditions

$$\frac{\partial}{\partial\rho}(U - A)|_{\lambda=0} = 0, \quad \frac{1}{\lambda}\frac{\partial}{\partial\lambda}(U - A)|_{\lambda=0} = 0.$$

Substituting these equations of expression U and A, we will obtain

$$6\pi c E\delta^{5/2}R^{1/2}\rho^2 - 2f\rho R = 0,$$

$$2\pi c E\delta^{5/2}R^{1/2}\rho^3\left(k^2 - \frac{(1+k^2)^2}{16}\right) + \frac{\pi\rho^2 E\delta^3}{3}(k^3 - k) - \frac{fR\rho^2 k^2}{2} = 0.$$

Multiplying the first equation on ρk^2, the second to 4 and deducting piecemeal, let us have

$$\pi c E\delta^{5/2}R^{1/2}\rho^3\frac{(k^2-1)^2}{2} + \frac{4}{3}\pi\rho^2 E\delta^3 k(1-k^2) = 0.$$

Hence

$$\rho = \frac{8}{3c}\frac{k}{k^2-1}\sqrt{\frac{\delta}{R}}.$$

Let us designate through

$$r = R\rho$$

a radius of the circle of bulge and will introduce the parameter

$$\varepsilon = \frac{R\delta}{r^2}.$$

Taking into account the obtained above value for ρ, we find the corresponding to it value ε:

$$\sqrt{\varepsilon} = \frac{3c(k^2-1)}{8k}.$$

Consequently, during the deformations, which satisfy the condition

$$\sqrt{\varepsilon} \geqslant \frac{3c(k^2-1)}{8k},$$

the bulge of shell in the form of circle is stable with respect to the disturbance/perturbations which are determined by parameter k.

One should, however, note that this conclusion we can draw only with respect to the disturbance/perturbations, which correspond to the small values of k, since $\sqrt{\varepsilon} > 1$ with large k. But the

determination of strain energy on the boundary of bulge assumes sufficient little of the parameter $\varepsilon (\varepsilon \ll 1)$.

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Experiment shows that the loss of stability of the axially symmetric form of bulge usually occurs with transition to star form with three apex/vertexes (k=3). Therefore we can count the form of the bulge of axially symmetric with

$$\rho \leqslant \frac{1}{c} \sqrt{\frac{\delta}{R}}.$$

For a radius of the circle of bulge $r=R\rho$, we will obtain

$$r \leqslant \frac{1}{c} \sqrt{R\delta}.$$

Let us find now force f by which the region of bulge begins to take the star form with three apex/vertexes (k=3). At the moment of transition to the star form of bulge in the state of the equilibrium of shell, the acting force f with the parameter of bulge ρ is connected by the relationship/ratio

$$6\pi c E \delta^{5/2} R^{1/2} \rho^2 \quad 2 f \rho R = 0.$$

We hence find value

$$f = \frac{3\pi E \delta^3}{R}.$$

When the acting force reaches this value, the region of bulge begins to take the star form with three apex/vertexes.

The obtained result about the stability of the axially symmetric

ones of the deformation of spherical shell with loading to concentrated force was subjected to experimental check. Tested spherical segment freely rested on steel ring (Fig. 15). The action of concentrated force of P in the form of the load of several steel washers was transferred through the vertical rod to the experience/tested shell. Shell was illuminated with the source of light S. Ghost image along parabolic line on the boundary of bulge was observed and was photographed.

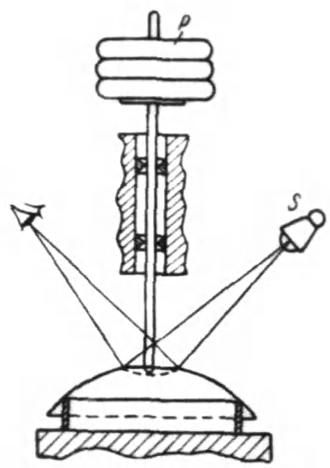

Fig. 15.

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During a gradual increase in load P, was noted the torque/moment when begins transition from bulge in the form of circle to the form of "triangle".

The results of experimental investigation are represented in Fig. 16. Here solid line depicts the obtained above dependence of the critical force P on the thickness of the shell

$$P = \frac{3\pi E\delta^3}{R}$$

for the copper shells of radius R=80 mm. Isolated points give the values of critical force for the shells of different thickness, above which was conducted the experiment. It is evident that the

experimental value of value P is close to its theoretical value.

Let us examine now a question concerning the stability of the axially symmetric deformation of spherical shell with loading by the uniform external pressure p.

Just as with the concentrated loading, at the moment of transition to the star form of bulge are satisfied the conditions

$$\frac{\partial}{\partial\rho}(U-A)\Big|_{\lambda=0}=0, \quad \frac{1}{\lambda}\frac{\partial}{\partial\lambda}(U-A)\Big|_{\lambda=0}=0,$$

where

$$A=\frac{\pi\rho^4R^3}{2}(1+\lambda^2k^2)\,p,$$

and U has previous value. Substituting under these conditions of value U and A, let us have

$$6\pi cE\delta^{5/2}R^{1/2}\rho^2-2\pi\rho^3R^3p=0,$$

$$2\pi cE\delta^{5/2}R^{1/2}\rho^3\left(k^2-\frac{(1+k^2)^2}{16}\right)+ \\ +\frac{\pi\rho^2E\delta^3}{3}(k^3-k)-\frac{\pi\rho^4R^3k^2p}{2}=0.$$

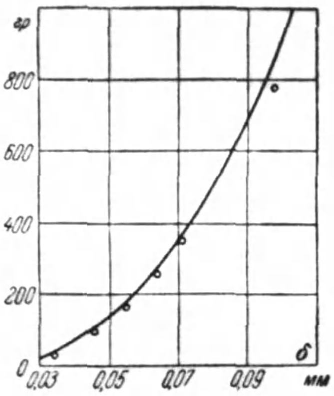

Fig. 16.

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We hence find the value ρ, which corresponds to ultimate strain

$$\rho=\frac{8}{3c}\frac{k}{k^2-1}\sqrt{\frac{\delta}{R}}.$$

It is interesting that this value ρ has accurately the same value, as in the case of the concentrated loading.

Designating, as before through $r=\rho/R$ a radius of the circle of bulge, let us have

$$r=\frac{8}{3c}\frac{k}{k^2-1}\sqrt{R\delta}.$$

Or, taking into account, that the transition to the star form of bulge occurs with k=3 (to star form with three apex/vertexes), we have

$$r=\frac{1}{c}\sqrt{R\delta}.$$

The obtained result about the stability of the axially symmetric deformation of spherical shell can be presented in the more foreseeable form, if we instead of radius r of the circle of bulge introduce sagging/deflection in center 2h. We have

$$2h = \frac{r^2}{R}.$$

Substituting here the critical value of r, determined on the formula

$$r = \frac{1}{c}\sqrt{R\delta},$$

we will obtain the critical sagging/deflection

$$2h = \frac{1}{c^2}\delta.$$

Hence

$$\frac{2h}{\delta} = \frac{1}{c^2} \simeq 28.$$

Thus, the supercritical deformation of flat spherical shell under the action of concentrated force or uniform external pressure is axially symmetric to those pores, while sagging/deflection 2h in the center of the region of bulge satisfies condition to the condition

$$\frac{2h}{\delta} \leqslant \frac{1}{c^2} \simeq 28.$$

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7. On lower critical load for flat spherical shell at uniform external pressure. In § 2 we obtained the row/series of the formulas, relating to the supercritical deformations of spherical shells. The application/use of these formulas was limited to the row/series of the conditions of the very common/general/total and indefinite

content, guaranteeing actually the axial symmetry of the supercritical deformations in question. Now, when the stability conditions of the axial symmetry of deformations we have explained, the region of applicability of the formulas indicated can be estimated more to determination.

For a flat spherical segment with curvature 1/R, radius of basis/base r and thickness δ under the assumption of the unlimited elasticity of the material of shell, was obtained (page 55) the following formula for the lower critical load:

$$p_l = 3cE\left(\frac{\delta}{R}\right)^2 \sqrt{\frac{\delta R}{r^2}}.$$

Since during

$$r \leqslant \frac{1}{c}\sqrt{R\delta}$$

deformation they are knowingly axially symmetric, with such r application/use of the formula indicated must not be limited by any conditions, including it is possible not to insist also on the special rigidity of the attachment of edge.

Thus, for flat, unlimitedly elastic spherical segments whose ratio of the height/altitude of segment h to thickness δ satisfies the condition

$$\frac{h}{\delta} \leqslant 14,$$

lower critical load at a uniform external pressure is determined from the formula

$$p_l = 3cE\left(\frac{\delta}{R}\right)^2 \sqrt{\frac{\delta R}{r^2}}.$$

Or, which is the same,

$$p_i = 3cE\left(\frac{\delta}{R}\right)^2 \sqrt{\frac{\delta}{2h}}.$$

For the spherical shells, which possess the limited elasticity, was derived (page 71) the following formula for a lower critical load at the external pressure:

$$p_i = 3cc'E\left(\frac{\delta}{R}\right)^3 \left(\frac{E}{\sigma_s}\right).$$

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This formula is obtained from that condition that the supercritical deformations limitedly elastic shells stop during appearance on the boundary of the bulge of plastic deformations from curvature, that is, from the condition

$$c'E\frac{r}{R}\left(\frac{\delta}{R}\right)^{1/2} = \sigma_s, \qquad (*)$$

where r - radius of the circle of bulge. Let us look, when determined by this condition deformation is axially symmetric. Substituting under condition (*) the critical value

$$r = \frac{1}{c}\sqrt{R\delta},$$

we will obtain

$$\frac{c'}{c}E\frac{\delta}{R} = \sigma_s.$$

Thus, the deformations in question will be axially symmetric, if

$$\frac{R}{\delta} < \frac{c'E}{c\sigma_s}.$$

Thus, the application/use of the formula

$$p_i = 3cc'E\left(\frac{\delta}{R}\right)^3 \left(\frac{E}{\sigma_s}\right)$$

for the lower critical load limitedly elastic spherical shells at an external pressure is limited to the condition

$$\frac{R}{\delta} < \frac{c'E}{c\sigma_{в}}.$$

Let us examine for an example the steel shell. Set/assuming

$$\sigma_{в} = 4 \cdot 10^3 \text{ кг/см}^2 \text{ (1)}, \quad E = 2 \cdot 10^6 \text{ кг/см}^2 \text{ (1)},$$
$$c' \simeq 0{,}9, \quad c \simeq 0{,}19,$$

Key: (1). kg/cm².

we will obtain

$$\frac{R}{\delta} < 2500.$$

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Chapter Two.

LOSS OF STABILITY OF STRICTLY CONVEX SHELLS.

Present chapter is dedicated to the analysis of stability of strictly convex hulls with the different methods of loading. The basis of our method of the study of problem compose the following two considerations:

1) the received by shell load at the moment of loss of stability is stationary and, therefore, little changes with the noticeable bulge of shell.

2) With considerable bulge the deformation of shell out of the vicinity of the boundary of the region of bulge can be considered geometric bending.

On the basis of these considerations, we will formulate and will base certain common/general/total principle, which then let us apply to the solution of specific problems. Specifically,, in § 1 we will examine the loss of stability of mildly sloping strictly convex

shells under external pressure, and in § 3 loss of stability of the shells of revolution. The principle indicated, let us call its "principle V", it is connected with principle A (chapter I, § 1) and reduces the study of the problem concerning the stability of shells to the solution of certain variational problem for the functional, determined on infinitesimal bending of the initial surface of shell.

§ 1. Loss of stability of strictly convex hulls under external pressure.

As noted above, the solution of the task of the stability of shells in our examination will be based on certain common/general/total principle. In present paragraph we will give the substantiation of this principle and will use it to the investigation of a question concerning stability of flat strictly convex hulls, which are located under external pressure.

1. Strain energy of shell. Let the elastic shell F be under the action of certain load q, which thus far more precisely formulate we will not. If load is small, then the elastic state of shell among the forms, close to F, it is determined unambiguously. Let us increase load q. Then can begin this torque/moment, when by the condition of nearness indicated the elastic state of shell is not unambiguously determined. Specifically,, together with the basic form of the

elastic equilibrium of the shell, for which the deformed surface of shell remains close to initial form (F), also, during further increase in the load there are other forms which are developed virtually without an increase in the effective load, moreover this development is accompanied by considerable changes in exterior form of shell. The load with which occurs the ambiguity of the elastic states of shell indicated, is called critical. The smallest critical load is called upper critical load.

The noted specific character of the elastic states of shell under the action of critical load (considerable changes in the form with stationary load) makes it possible to reduce the problem of the determination of such loads to the examination of the supercritical states of shell and to use the methods, developed in chapter 1, to the study of these states.

Let the loss of stability of shell under the action of the given load be accompanied by the bulge of region G, limited by curve γ. On the basis of the demonstrative representation of the character of bulge, we assume that essential deformations the shell experience/tests only in the vicinity of boundary of the region G, and out of this vicinity the form of the deformed shell is close to initial. In this case, it is logical to consider that energy of the deformed shell is concentrated in the vicinity indicated. For

determining this energy, we will use the same considerations, as in the examination of substantially supercritical deformations in § 1 chapter 1.

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Identifying the isometric conversion of initial surface with this quite surface and reproducing the reasonings of §1 of chapter 1, we will obtain the same in form as there, expression for $\bar{U}_\gamma$ - strain energy of shell per the unit of length γ (boundary of the region G). Specifically,,

$$\bar{U}=\frac{D}{2}\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} v''^2\,ds+\frac{D'}{2}\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}}\frac{u^2}{\rho^2}\,ds.$$

The corresponding formula (page 29) for substantially supercritical deformations contained two additional term/component/addends

$$D\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}}\Delta k v''\,ds+D\bar{\varepsilon}\,\Delta k^2.$$

In this case these term/component/addends are equal to zero, since the isometric conversion F into $\tilde{F}$ is identical and, therefore, Δk=0. Let us recall that in formula for U_γ value u and v designate the displacement (during deformation) respectively in the direction of the principal normal and binormal of curve γ of that point of surface to which $\bar{U}_\gamma$ is related, and through D and D' are designated the

rigidity of shell to curvature and elongation - compression respectively. Displacements u, v are connected by the relationship/ratio

$$u' + \alpha v' + \frac{v'^2}{2} = 0,$$

where α - an angle between the osculating plane curved γ and tangential plane of surface.

Just as in § 1 of chapter 1, instead of the variables u, v, s we introduce new the variables $\bar{u}$, $\bar{v}$, $\bar{s}$ according to the formulas

$$\bar{u} = \frac{u}{\varepsilon\rho\alpha^2}, \quad \bar{v} = \frac{v'}{\alpha}, \quad \bar{s} = \frac{s}{\rho\varepsilon},$$

$$\varepsilon^4 = \frac{\delta^2}{12\rho^2\alpha^2}.$$

Here ρ - radius of curvature curved γ, δ - thickness of shell.

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In the new variables the feature above which for simplicity of recording let us lower, we will obtain

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{2\cdot 12^{3/4}(1-\nu^2)} \int_{-\bar{\varepsilon}^*}^{\bar{\varepsilon}^*} (v'^2 + u^2)\, ds.$$

Integration limits $\bar{\varepsilon}^*$ and $-\bar{\varepsilon}^*$ unlimitedly increase in the absolute value together with $\rho\alpha/\delta$. Therefore, being limited to the case of such shells and deformations, for which $\delta/\rho\alpha$ is small, integration limits can be replaced by $\pm\infty$. Then

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{2\cdot 12^{3/4}(1-\nu^2)} \int_{-\infty}^{\infty} (v'^2 + u^2)\, ds.$$

As always, let us assume the symmetry of function $\check{v}(s)$ and antisymmetry u(s). Then it is possible to be restricted to integration in limits $(0, \infty)$. Therefore

$$\bar{U} = \frac{E\delta^{5/2}a^{5/2}p^{-1/2}}{12^{3/4}(1-\nu^2)} \int_0^\infty (v'^2 + u^2)\,ds.$$

Let us agree to designate that part of region G, which is arrange/located out of the vicinity in question by curve γ, through A_1, vicinity itself - through A_{12}, and the remaining part of the shell - through A_2. The found by us expression for energy $\bar{U}$ depends substantially on the form of shell in transition zone A_{12}, which (form) is determined by functions u, V, which assign deformation. So as during the investigation of supercritical deformations in chapter 1, energy $\bar{U}$ we will determine from the condition of the minimum during the assigned/prescribed common/general/total deformation. This deformation we characterize by sagging/deflection h in the region of extrusion near this point of curve γ, to which (to point) energy $\bar{U}$ is related. All this acquires the precise sense when the width of transition zone A_{12} unlimitedly decreases.

In the initial variables $\check{v}$ and s, value h allow/assumes the obvious representation

$$h = -\int_{-\varepsilon^*}^{\varepsilon^*} v'\,ds.$$

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If we pass to new variables and integration limits $\bar{\varepsilon}^*$ and $-\bar{\varepsilon}^*$ replace by $+\infty$, then we will obtain

$$h = -\frac{1}{12^{1/4}} \sqrt{\delta\rho\alpha} \int_{-\infty}^{\infty} v\,ds$$

or, taking into account the predicted symmetry of function $\check{v}(s)$,

$$h = -\frac{2}{12^{1/4}} \sqrt{\delta\rho\alpha} \int_{0}^{\infty} v\,ds.$$

Thus, energies $\bar{U}$ and, consequently, also functions u, v, on which it depends, they are determined from the condition of the minimum of the functional

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\nu^2)} \int_{0}^{\infty} (v'^2 + u^2)\,ds$$

during the additional limitation

$$-\frac{2}{12^{1/4}} \sqrt{\delta\rho\alpha} \int_{0}^{\infty} v\,ds = h = \text{const}.$$

The varied functions u, v, besides integral communication/connection indicated, satisfy another relationship/ratio

$$u' + v + \frac{v^2}{2} = 0 \qquad (*)$$

and they turn into zero at infinity.

Let us examine the task of the minimum of functional $\bar{U}$. In connection with this we, first of all, convert communication/connection

$$-\frac{2\sqrt{\delta\rho\alpha}}{12^{1/4}} \int_{0}^{\infty} v\,ds = h$$

with the aid of relationship/ratio (*). If this relationship/ratio is integrated within the limits (-∞, ∞) and to considered in this case that u(-∞)=u(∞)=0, then we will obtain

$$-\int_{-\infty}^{\infty} v\,ds = \int_{-\infty}^{\infty} \frac{v^2}{2}\,ds.$$

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Hence, taking into account the symmetry of function $\check{\Psi}$(s), we will obtain

$$-\int_{0}^{\infty} v\,ds = \int_{0}^{\infty} \frac{v^2}{2}\,ds.$$

Consequently, integral communication/connection, to which is subordinated function $\check{\Psi}$(s), can be represented in the form

$$\frac{\sqrt{\delta\rho\alpha}}{12^{1/4}} \int_{0}^{\infty} v^2\,ds = h.$$

Thus, our variational problem consists of the determination of the minimum of the functional

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{3/2}\rho^{-1/2}}{12^{3/4}(1-\nu^2)} \int_{0}^{\infty} (v'^2 + u^2)\,ds$$

under the conditions

$$\frac{\sqrt{\delta\rho\alpha}}{12^{1/4}} \int_{0}^{\infty} v^2\,ds = h = \mathrm{const},$$

$$u' + v + \frac{v^2}{2} = 0, \qquad (*)$$

$$u(0) = u(\infty) = v(\infty) = 0.$$

Since us interests the initial stage of supercritical

deformation, in relationship/ratio (*) term/component/addend $v^2/2$ can be disregarded, after giving, thus, to this relationship/ratio the entirely simple form

$$u' + v = 0.$$

If we now everywhere replace v by -u', then we come to the task of the minimum of the functional

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\nu^2)}\int_0^\infty (u''^2 + u^2)\,ds$$

with integral communication/connection

$$\frac{\sqrt{\delta\rho\alpha}}{12^{1/4}}\int_0^\infty u'^2\,ds = h = \text{const}$$

and the boundary conditions for the varied function

$$u(0) = u(\infty) = 0.$$

According to Eylera - Lagrange's method, our variational problem is reduced to the examination of the unconditional extremum of the functional

$$W = \int_0^\infty \left\{ \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\nu^2)}(u^2 + u''^2) - \lambda\frac{\sqrt{\delta\rho\alpha}}{12^{1/4}}u'^2 \right\} ds,$$

where λ - certain constant.

Set/assuming for the brevity

$$\sigma = \frac{\sqrt{12}(1-\nu^2)\rho\lambda}{E\alpha^2\delta^2},$$

we can consider that the discussion deals with the extremum of the functional

$$J = \int_0^\infty (u^2 + u''^2 - \sigma u'^2)\,ds,$$

which differs from W only in terms of constant factor.

The equation of eylera - Lagrange for functional J will be

$$u^{IV} + u + \sigma u'' = 0.$$

Its general solution -

$$u(s) = \sum c_k e^{\omega_k s},$$

where ω_k - roots of the characteristic equation

$$\omega^4 + 1 + \sigma\omega^2 = 0.$$

In order to satisfy boundary condition $u(\infty)=0$, it is necessary that among the characteristic roots ω_k there would be two roots with negative real part. If these roots are designated ω_1 and ω_2, then the solution of our variational problem it gives function u(s) of the form

$$u = c_1 e^{\omega_1 s} + c_2 e^{\omega_2 s}.$$

In order to satisfy and to the second boundary condition

$$u(0) = 0,$$

it is necessary to require that

$$c_1 = -c_2 = c.$$

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In this case, for function u(s) is obtained the expression

$$u = c(e^{\omega_1 s} - e^{\omega_2 s}).$$

Let us substitute the obtained function in the expression of functional $\overline{U}$ and in communication/connection. Then we obtain

$$\int_0^\infty u'^2\,ds = -c^2\left(\frac{\omega_1}{2}+\frac{\omega_2}{2}-\frac{2\omega_1\omega_2}{\omega_1+\omega_2}\right),$$

$$\int_0^\infty u^2\,ds = -c^2\left(\frac{1}{2\omega_1}+\frac{1}{2\omega_2}-\frac{2}{\omega_1+\omega_2}\right),$$

$$\int_0^\infty u''^2\,ds = -c^2\left(\frac{\omega_1^3}{2}+\frac{\omega_2^3}{2}-\frac{2\omega_1^2\omega_2^2}{\omega_1+\omega_2}\right).$$

Noting that in our case the roots ω_1 and ω_2 are conjugate/combined in pairs and equal to unity in absolute value, we can write

$$\omega_1 = e^{i\vartheta}, \quad \omega_2 = e^{-i\vartheta}.$$

The substitution of these values into our integrals gives

$$\int_0^\infty u'^2\,ds = c^2\frac{\sin^2\vartheta}{\cos\vartheta},$$

$$\int_0^\infty (u^2+u''^2)\,ds = c^2\frac{\sin^2\vartheta}{\cos\vartheta}(2+4\cos^2\vartheta).$$

Hence

$$\bar{U} = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\nu^2)}c^2\frac{\sin^2\vartheta}{\cos\vartheta}(2+4\cos^2\vartheta),$$

$$h = \frac{\sqrt{\delta\rho\alpha}}{12^{1/4}}c^2\frac{\sin^2\vartheta}{\cos\vartheta}.$$

Consequently,

$$\bar{U} = \frac{E\delta^2\alpha^2 h}{\sqrt{12}(1-\nu^2)\rho}(2+4\cos^2\vartheta).$$

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Minimum $\bar{U}$ at h=const is reached with $\vartheta=\pi/2$. And we obtain the following resultant expression for strain energy:

$$\bar{U} = \frac{2E\delta^2\alpha^2 h}{\sqrt{12}(1-\nu^2)\rho}.$$

2. Formulation of principle V and its substantiation. Let the loss of stability of shell under the action of the given load be accompanied by the bulge of region G, limited by curve γ. Just as in the examination of substantially supercritical deformations in chapter 1, let us approach the form of shell after loss of stability and noticeable bulge the isometric conversion of initial surface. Taking into account, that essential deformations the shell experience/tests only in the vicinity of the boundary of bulge, we visualize, that this isometric conversion is accompanied by the appearance of two fin/edges γ' and γ'', close to γ(Fig. 17). If γ' and γ'' they are plane curves, then an example of this isometric conversion is the mirror reflection of the region, limited by curve γ'', with the subsequent mirror reflection of its part, limited by curve γ'. In order to reduce designations, let us suppose that curve γ'' coincides with γ. Furthermore, for certainty let us consider that the curve γ' is arrange/located within region G and limits region G'.

Let us examine the task of the isometric conversion of surface of F with the extrusion of region G' and the formation of fin/edges along curves γ and γ'.

Certain indeterminancy/uncertainty of this task is removed by

requirement that unknown surface within region G' and out of region G was equally oriented with F, but in region G out of region G', it had opposite orientation.

If curve γ' coincides with γ, then task has the trivial solution.

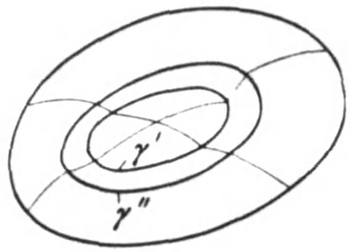

Fig. 17.

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The isometrically converted surface is very surface of F. It is logical to assume that with sufficient nearness of curves γ and γ' the isometric conversion of surface of F is characterized by essential deformations only in the band between these curves. As far as part is concerned remaining of the surface, here final bending can be replaced infinitesimal.

Keeping in mind concrete/specific/actual application/appendices, we will not explain the structure of the converted surface in the band between curves γ and γ', since in actuality deformation in this band for an elastic shell is determined by energy considerations. The bending of the band indicated we will describe by certain common/general/total property which will allow us to explain the conditions of the coupling infinitesimal bending out of region G and

within region G' upon transfer to the limit:

$$\gamma' \to \gamma.$$

P - arbitrary point to curve γ. Let us conduct from this point inside region G geodetic perpendicular before intersection with curve γ' at point P'. s - length of this perpendicular. Upon transfer from surface of F to the isometrically converted surface of point P and P', will be obtained the displacement τ_P and $\tau'_{P'}$, where τ and τ' they designate the which bend surface fields of F in the appropriate regions. Let us determine difference $\tau_P - \tau'_{P'}$, assuming sufficient nearness of curves γ and γ'.

The formation of fin/edge according to line γ during the isometric conversion of surface of F is accompanied by the rotation of tangential plane about tangent to curve γ. Upon transfer to limit $\gamma' \to \gamma$, this rotation is reduced to mirror reflection in osculating plane by curve γ. Hence it follows that with sufficient nearness of curves γ and γ' the vector $\tau_P - \tau'_{P'}$ can be considered directed perpendicularly to osculating plane by curve γ, that is along the binormal of this curve. Then

$$\tau_P - \tau'_{P'} = \sigma e,$$

where e - the unit vector of binormal. As far as factor is concerned σ, at small angle α between the osculating plane to curve γ and the tangential planes of surface it is equal to $2\alpha s$.

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As noted above, in connection with the forthcoming application/appendices us it will interest the case of close curves γ' and γ. In connection with this we pass to limit $\gamma' \to \gamma$. Now the task of the bending of surface F lies in the fact that to find the fields infinitesimal bending - field τ' within region G and τ - out of this region which on overall boundary of the region γ satisfy the condition

$$\tau - \tau' = \sigma e. \qquad (*)$$

Here e - unit vector of binormal curved γ, and σ - certain function, assign/prescribed in this curve. Condition (*) we will call the condition of coupling.

Knowing the bending fields τ and τ', we know the form of the deformed shell out of the vicinity of the boundary of bulge γ, therefore, we can determine strain energy U and produced by external load work A. Really/actually, in the expression of strain energy $\bar{U}$ (p. 1) is contained value h - sagging/deflection in the region of bulge near boundary γ (more precise, a change in the sagging/deflection upon transfer through the boundary). This sagging/deflection there is nothing else but value σ, entering under the condition of the coupling of fields τ and τ'. Thus, it is possible to count that the strain energy is determined by the bending

fields. Further, since the bending fields determine in essence the form of the deformed shell, with the assigned/prescribed load they determine produced by this load work by deformation.

We see that to the state of the elastic equilibrium of shell with bulge under the action of critical load corresponds the field infinitesimal bending with breakage along some lines, which, as usual, communicates to the functional

$$W=U-A$$

steady-state value.

Thus, we come to which follows principle B.

If the effective on shell load critical, then variational problem for the functional

$$W=U-A$$

on the disruptive infinitesimal bending of median surface of shell has the nontrivial solution, that is the bending field, which is the solution, not is equal to zero identically.

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Functional W is determined in the fields infinitesimal bending with the breakage, which satisfy the condition

$$\tau-\tau'=\sigma e.$$

where $\tau - \tau'$ - breakage of the bending field, and e - unit vector of the binormal curved γ, along which occurs the breakage.

Addend U of functional W is determined from the formula

$$U = \int\limits_{\gamma} \frac{2E\,\delta^2\alpha^2\sigma}{\sqrt{12}\,(1-\nu^2)\,\rho}\,ds.$$

Here ρ - radius of curvature curved γ, where occurs the breakage of the bending field; α - angle between the osculating plane curved γ and tangential plane of surface; σ - component of the breakage of the bending field in binormal curved γ, δ - the thickness of shell, E - the module/modulus of elasticity ν— Poisson ratio. Integration is fulfilled according to arc s by curve γ.

Addend A of functional W is defined by the usual method as the produced by external load work by the deformation, given by the bending field.

The solution of task regarding critical load with the aid of principle B is conjugate/combined with known difficulties, since we know either lines of discontinuity for the bending field or character of breakage (function σ). However, in specific problems we can obtain the appropriate information from experimental data and thus remove the difficulty indicated.

In conclusion we want to pay attention to one more fact, connected with the application/use of principle B. If the effective on shell load is concentrated along certain line and loss of stability occurs so that this line proves to be in the vicinity of the boundary of bulge, then, by determining work A, produced by load, it is necessary to consider the form of the deformed shell in the vicinity indicated. As this one should make, we will show in p. 5 based on the specific example of the loss of stability of slightly curved shell of rotation under the action of the normal external pressure, evenly distributed along certain parallel.

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3. Determination of critical load for strictly convex hulls at uniform external pressure. Let the strictly convex shell, rigidly attached on edge, be under the external pressure p. When pressure reaches critical value, shell loses stability and it begins to be swelled. Let G - the region of bulge and γ - limiting it is curved. According to principle B, with critical loading the variational problem for the functional

$$W = U - A,$$

determined on infinitesimal bending with discontinuous change

lengthwise γ, has the nontrivial solution. We will use this property of critical load for its determination.

First of all, we will restrict the class of the bending fields, in which is examined functional W. For this, we will use rigidity condition of the attachment of shell along edge.

τ' - bending field out of region G, and τ - bending field within this region. On the boundary γ of region G of field τ and τ', they satisfy the condition of the coupling

$$\tau - \tau' = \sigma e,$$

where e - the unit vector of binormal curved γ. In view of the rigidity of the attachment of edge of surface, we must consider field τ' at edge of surface equal to zero. But then according to known theorem it is equal to zero on an entire surface, i.e., out of region G. Hence, taking into account the condition of the coupling of fields τ' and τ, we consist that to curve γ

$$\tau = \sigma e.$$

Is known the theorem according to which satisfying this condition bending field τ either is equal to zero identically, if curved γ not flat/plane [1] or it is the velocity field of the motion of surface as whole, if is curved γ flat/plane.

FOOTNOTE [1]. Generally speaking, in that form, as it is here

formulated, this theorem for twisted curve γ inaccurate. But, on the basis of the demonstrative picture of deformation, it is possible to count that curve γ differs little from plane curve, and that $\sigma \simeq \text{const.}$ but under such conditions the corresponding infinitesimal bending differ little from trivial ones. (Editor's notes). ENDFOOTNOTE.

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In the latter case this motion is rotation about the straight line, which lies at plane curved γ; in particular, this can be shift/shear in the direction, perpendicular to this plane.

Thus, the rigidity of the attachment of edge of surface predetermines the form of the region of bulge and the character of the bending fields. Specifically,, the region of bulge is limited by plane curve, moreover out of this region the bending field $\mathbf{r}'=0$, and within the region the bending field $\boldsymbol{\tau}=\mathbf{a}\times\mathbf{r}+\mathbf{b}$, where r — a radius-vector of surface, a and b - constant vectors.

Let us note that the approach/approximation of the form of the deformed shell infinitesimal isometric conversion which we obtained, there is actually approach/approximation by twofold mirror reflection. Here the role of curves γ' and γ'', close to γ, play plane curves. Isometric conversion is of the mirror reflection of the

region, limited by curve γ', in the plane of this curve, and then the mirror reflection of its part, limited by curve γ'', in plane the latter (Fig. 18).

Let us introduce the system of Cartesian coordinates, after accepting the tangential plane of surface, parallel to plane as curve γ, for plane xy, and point of contact of tangency P - in the origin of coordinates. Of the axes of coordinates x and y is directed along the main directions of surface at point P. Since field $\tau = a \times r + b$, then for its component ζ along the axis z we will obtain

$$\zeta = c_1 x + c_2 y + h,$$

where x, y -- coordinates of the point, at which is calculated this component, and c_1, c_2 and h - constants.

Let us assume now that the region of bulge is small. In this case it has a form of the ellipse, similar to the indicatrix of curvature at point P, and its boundary γ can be assign/prescribed by the equations

$$x = \lambda \sqrt{R_1} \cos t, \quad y = \lambda \sqrt{R_2} \sin t.$$

Here R_1 and R_2 - main radii of curvature in the center of bulge P, and λ - parameter, which characterizes the size/dimensions of the region of bulge.

Fig. 18.

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Let us calculate the strain energy of shell. We have (page 109)

$$U = \int_{\gamma} \frac{2E\,\delta^2\alpha^2\zeta}{\sqrt{12}\,(1-\nu^2)\,\rho}\,ds.$$

Here α - **angle between the plane curved γ and tangential plane** of surface by lengthwise curve, ρ - radius of curvature curved γ, and ζ - sagging/deflection in the region of bulge by lengthwise curve γ.

Let us determine values α and ρ, entering the formula for strain energy U. **We have**

$$\frac{1}{\rho} = \frac{\sqrt{R_1R_2}}{\lambda\,(R_1\sin^2 t + R_2\cos^2 t)^{3/2}}.$$

On Meusnier's formula the angle

$$\alpha \simeq \rho k_n,$$

where k_n - normal surface curvature of shell in the direction of **tangential curve γ. On the** Euler formula the curvature

$$k_n = \frac{1}{R_1}\left(\frac{R_1\sin^2 t}{R_1\sin^2 t + R_2\cos^2 t}\right) + \frac{1}{R_2}\left(\frac{R_2\cos^2 t}{R_1\sin^2 t + R_2\cos^2 t}\right),$$

i.e.,

$$k_n = \frac{1}{R_1\sin^2 t + R_2\cos^2 t}.$$

The cell/element of arc to curve γ is equal to

$$ds = \lambda (R_1 \sin^2 t + R_2 \cos^2 t)^{1/2} dt.$$

Substituting the obtained values in formula for U, we will obtain

$$U = \int_0^{2\pi} \frac{2E\,\delta^2 \zeta \lambda^2\, dt}{\sqrt{12}\,(1-\nu^2)\sqrt{R_1 R_2}},$$

i.e.,

$$U = \frac{4\pi E\,\delta^2 h \lambda^2}{\sqrt{12}\,(1-\nu^2)\sqrt{R_1 R_2}}.$$

Let us calculate now the work A produced by external load. We have

$$A = Qh.$$

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Here Q - total load, acting on the region of bulge, and h - sagging in the center of bulge P. It is obvious,

$$Q = pS,$$

where S - area of the region of bulge, and p - pressure. In the case in question area S as the area of ellipse with semi-axes $\lambda\sqrt{R_1}$ and $\lambda\sqrt{R_2}$, is equal to

$$\pi\lambda^2\sqrt{R_1R_2}.$$

Consequently,

$$A = Qh = pSh = \pi p\sqrt{R_1R_2}\,h\lambda^2.$$

Now from the condition of the equilibrium

$$d(U - A) = 0$$

we find the received by shell load. We have

$$d\left\{\frac{4\pi E\,\delta^2 h\lambda^2}{\sqrt{12}\,(1-\nu^2)\sqrt{R_1R_2}} - \pi p\sqrt{R_1R_2}\,h\lambda^2\right\} = 0.$$

whence

$$p = \frac{2E}{\sqrt{3}\,(1-\nu^2)}\,\frac{\delta^2}{R_1R_2}.$$

As one would expect, load p was stationary with respect to parameter $h\lambda^2$, characterizing bulge.

Thus, upper critical pressure p_e for the flat strictly convex

hull, rigidly attached on edge, is determined from the formula

$$p_e = \frac{2E}{\sqrt{3}\,(1-\nu^2)} \frac{\delta^2}{R_1 R_2},$$

where R_1, R_2 - main radii of curvature of shell, δ - its thickness, E and ν - module/modulus of elasticity and Poisson ratio.

Let us note that $1/R_1R_2$ there is Gaussian curvature. Therefore formula can be written also in this form:

$$p_e = \frac{2E\,\delta^2 K}{\sqrt{3}\,(1-\nu^2)},$$

where K - Gaussian curvature of median surface of shell.

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For the spherical shell of radius R, we have

$$R_1 = R_2 = R,$$

and formula for the value of critical pressure takes the form

$$p_e = \frac{2E}{\sqrt{3}\,(1-\nu^2)} \left(\frac{\delta}{R}\right)^2.$$

In view of the fact that

$$1-\nu^2 \simeq 1,$$

this formula gives actually well known result for the spherical shells

$$p_e = \frac{2E}{\sqrt{3(1-\nu^2)}} \left(\frac{\delta}{R}\right)^2.$$

The derived study of the problem concerning critical load at a uniform external pressure on slightly curved shell substantially rests on assumption about the rigidity of the attachment of edge. In

§3 we again will turn to this question. Without assuming the special rigidity of the attachment of edge, we will proceed from some natural assumptions about the character of bulge, prompted by experiment. It is interesting that the critical pressure in this case will be obtained the same.

From the formula, which is determining the value of the received by shell pressure with bulge, we see that this value does not depend on parameter $h\lambda^2$, which characterizes deformation, in particular, from the size/dimension of the region of bulge (parameter λ). It is hence logical to draw the conclusion that at the nonuniform, but slowly changing external pressure on shell the critical load is determined by the value of maximum pressure.

4. Effect of initial bending of shell on stability. The determination of critical load for strictly convex hulls, which are located under external pressure, was the object/subject of numerous experimental experiments. They experimented usually with spherical shells. The results of these investigations, as a rule, did not give the specific critical pressure. Is singular, which constant/invariably was observed this the fact that the critical pressure, obtained in experiment, is always less than theoretical value p_e, determined on the formula

$$p_e = \frac{2E}{\sqrt{3}\,(1-\nu^2)}\left(\frac{\delta}{R}\right)^2.$$

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The basic reason, which lowers the value of critical pressure, is the imperfection of the form of real shell or, as they say, initial bending. The thoroughly placed experiments with the accurately prepared spherical segments show that the theoretical value of critical pressure is realized. Thus, for instance, in the experiment regarding lower critical load, which is described in §2 of chapter 1, was determined also the upper critical pressure by which the shell lost stability.

Figure 19 gives the curve of the dependence of the upper critical pressure

$$p_e=\frac{2E}{\sqrt{3}\,(1-\nu^2)}\left(\frac{\delta}{R}\right)^2$$

on thickness δ for the copper shells of radius R=80 mm. Isolated points give the critical value of pressure, obtained in the described experiment. For simplicity the modulus of elasticity is accepted equal to $1\cdot10^6$ kg/cm². If we consider the actual value of module/modulus E, then theoretical curve virtually traverses experimental points.

The fact that the true critical pressure for the real shell,

which has initial bending, can be considerably lower than theoretical value, causes serious difficulties during the design of shells, since this value cannot be accepted for calculated. As natural output/yield from this position it would be accept as design load lower critical load. This load is determined by considerable deformations and therefore it is less sensitive to the imperfections of the form of shell. If we accept lower critical load for calculated, then the loss of stability of shell completely is eliminated, since the received by shell load during supercritical deformation is more than lower critical.

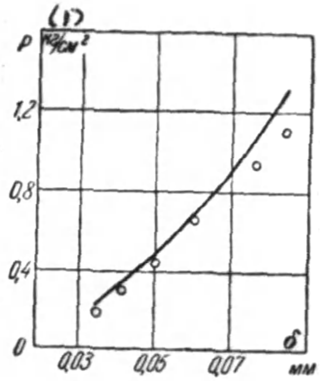

Fig. 19.

Key: (1). kg/cm².

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The solution of a question indicated concerning design load is simple and reliable. However, it cannot be accepted because of the very low value of lower critical load. Let us examine this based on the example of the shell, which has the form of flat spherical segment, which is located under external pressure. For it upper critical pressure p_e is determined from the formula

$$p_e = \frac{2E}{\sqrt{3}\,(1-\nu^2)}\left(\frac{\delta}{R}\right)^2,$$

where R - a radius of curvature of shell, and δ - its thickness. As far as pressure is concerned lower critical, it is equal

$$p_i = 3cE\left(\frac{\delta}{R}\right)^2\sqrt{\frac{\delta}{2h}},$$

where h - a height/altitude of segment, and the constant c≈0.19.

$$\frac{p_i}{p_e} \simeq 0{,}5\sqrt{\frac{\delta}{2h}}.$$

Hence

We see that already with h=8δ $p_l/p_e \simeq 0.1$ i.e., the lower critical value composes 0.1 from upper.

The acceptable solution of a question concerning design load would be definition it as the load, with which occurs the loss of stability taking into account initial bending. Now we will make the attempt to determine this load in connection with the load case of flat strictly convex hull by external pressure.

In §2 of chapter 1 for the value of the received by shell load p with bulge on height/altitude 2h, is obtained the formula

$$p = 3cEH\sqrt{K}\,\delta^2\sqrt{\frac{\delta}{2h}}, \qquad (*)$$

where K - Gaussian, and H - the mean curvature of shell. It is logical to assume that when, in the shell, initial bending is present, which corresponds to the form of bulge, it will lose stability at the pressure which is determined by this formula. Thus, during initial bending 2h we propose to determine the critical pressure p according to formula (*). considering it calculated.

It must be noted that formula (*) have derived we under the assumption of sufficient smallness of the parameter δ/2h. Therefore its to apply is possible only in the presence of considerable initial

bending.

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For a spherical shell formula (*) takes the form

$$p = 3cE\left(\frac{\delta}{R}\right)^2 \sqrt{\frac{\delta}{2h}} \equiv kE\left(\frac{\delta}{R}\right)^2.$$

Accompanying graph (Figure 20) depicts the dependence of the dimensionless coefficient k as a function of initial bending 2h/δ.

From this graph it is evident that the high precision of manufacturing shell is hardly justified by orientation at the values of critical pressure, close to theoretical.

5. Loss of stability of shell and critical loads in other load cases by external pressure. As shown above, obtained by us result about loss of stability with the loading of strictly convex hull by external pressure is related not only to the case of uniform pressure. To a certain extent by it it is possible to use, determining critical load at a continuous, compulsorily constant pressure. Now we want to examine other cases where the continuity condition of load distribution according to the surface of shell substantially is not satisfied. Among these cases basic are the load distribution along certain line and the loading, concentrated at point.

A question concerning the concentrated loading of strictly convex hull by us has already been examined in chapter 1 (p. 2 §2), where it is shown, that this loading does not lead to loss of stability. Thus, for the completeness of the investigation of a question to us it remains to examine the load case, distributed along certain line on the surface of shell. It is not difficult to give the example where this loading is realized. We bear in mind the loading of shell by pressure the tightly drawn on it filament. Let us examine the task of loss of stability and critical load in this concrete/specific/actual setting.

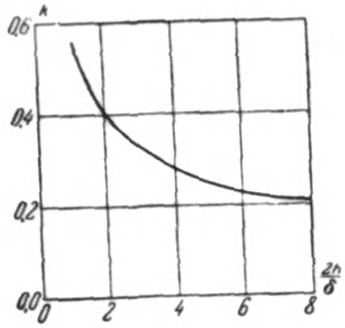

Fig. 20.

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Thus, let the strictly convex hull, rigidly attached on edge, experience/test pressure the tightly drawn on it filament (Fig. 21a). With certain tension of filament Q, the shell loses stability with the education/formation of the regions of bulge along the line of the adjoining of filament (Fig. 21b). Let us determine the value of this critical tension.

In contrast to the examined case of pressure, distributed over the surface where the loss of stability is accompanied by the bulge immediately of certain finite domain, in this case the bulge spreads from certain center on the line of the adjoining of filament to shell.

It is natural to approach the form of shell during supercritical

deformation with the aid of simple mirror bulge, as this was made in chapter 1 (§2). In this case, for energy U of elastic deformation of shell, is obtained the expression (page 49)

$$U = \pi c E (2h)^{3/2} \delta^{5/2} (k_1 + k_2).$$

Here 2h - sagging/deflection in the center of bulge, k_1 and k_2 - principal curvatures of shell, δ - thickness, E - the module/modulus of elasticity, but is constant c=0.19.

Produced by the tension of filament Q work is equal to

$$A = Q \Delta l,$$

where Δl - common/general/total displacement of the ends of the filament, connected with the bulge of shell.

Assuming that friction between the filament and the shell is absent, and therefore filament fits closely to shell according to certain arc AB of geodetic line, easily we find Δl. It is equal to the difference between arc AB and chord, connecting the ends. If we designate the normal curvature of surface of shell in the direction of the filament through k_n, and sagging/deflection in the center of the bulge through 2h, then

$$\Delta l = \frac{1}{3} (2h)^{3/2} \sqrt{k_n}. \qquad (*)$$

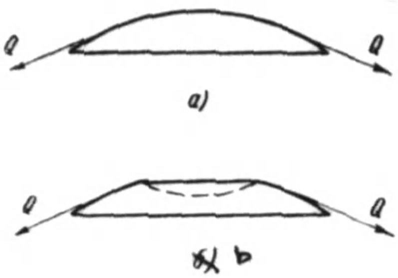

Fig. 21.

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Really/actually, if we introduce the Cartesian coordinates x, y, z, after accepting the osculating plane of filament in the center of bulge for plane xy, and tangent for x axis, then is filament, fitting closely to surface on geodetic, it will be assigned by the equations

$$y=\frac{k_n x^2}{2}+O(x^3), \quad z=O(x^3),$$

where through $O(x^3)$ designated are the values, which are of the order x^3.

If chord AB has length 2d, then arc length AB is equal to

$$s=\int_{-d}^{d}\sqrt{1+k_n^2x^2}\,dx \simeq 2d+\frac{k_n^2d^3}{3}.$$

Or, noting that $2h \simeq k_n d^2$, we get

$$s=2d+\frac{1}{3}(2h)^{3/2}\sqrt{k_n}.$$

Hence is obtained indicated on formula (*) value Δl.

Substituting the value Δl in the expression of work A, we will

obtain

$$A = \frac{1}{3} Q (2h)^{3/2} \sqrt{k_n}.$$

Now from the condition of the equilibrium of the shell

$$d(U - A) = 0,$$

where is varied sagging/deflection 2h, we find the received by shell load Q. We have

$$d\left\{\pi c E (2h)^{3/2} \delta^{5/2} (k_1 + k_2) - \frac{1}{3} Q (2h)^{3/2} \sqrt{k_n}\right\} = 0.$$

Hence

$$Q = 3\pi c E \delta^{5/2} (k_1 + k_2) \frac{1}{\sqrt{k_n}}.$$

We see that as in the examined case of the continuous loading of the surface of shell, tension Q is stationary with respect to the parameter 2h, which characterizes bulge.

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Thus, critical tension Q_e of filament, by which the shell can lose stability and begin to be swelled, is determined from the formula

$$Q_e = 3\pi c E \delta^{5/2} (k_1 + k_2) \frac{1}{\sqrt{k_n}}.$$

For the spherical shell of radius R, we have

$$k_1 = k_2 = \frac{1}{R},$$

and formula for the value of critical tension takes the form

$$Q_e = 6\pi c E \delta^2 \sqrt{\frac{\delta}{R}}.$$

Let us examine a question concerning the loss of stability of the flat conical shell of revolution. Let the flat, rigidly attached on edge conical shell under the uniform external pressure p lose stability, having the axial symmetry of bulge (Fig. 22). Let us determine the value of critical load.

ρ - radius of the circle of bulge γ, and α - angle between the plane of basis/base and generatrix. Then the strain energy of shell (with bulge) per the unit of the length of circumference γ will be (page 109)

$$\widetilde{U}=\frac{2E\delta^2\alpha^2h}{\sqrt{12}(1-\nu^2)\rho}.$$

where h - sagging/deflection on the region of bulge. Total energy of the deformation of shell will be equal to

$$U=\frac{4\pi E\delta^2\alpha^2h}{\sqrt{12}(1-\nu^2)}.$$

The produced by external pressure work by the bulge of shell is equal to

$$A=\pi\rho^2hp.$$

where p - the critical pressure under action of which occurs the bulge.

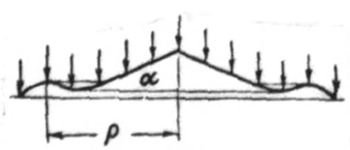

Fig. 22.

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From the condition of the elastic equilibrium of shell with the bulge

$$\frac{d}{dh}(U-A)=0$$

we obtain the value of the critical pressure

$$p=\frac{2E\delta^2\alpha^2}{\sqrt{3}\,(1-\nu^2)\,\rho^2}.$$

The minimum value p, i.e., upper critical load, is obtained with maximum ρ. As this value ρ, it is necessary to take a radius of the basis/base of shell r.

Thus, upper critical load for the flat rigidly attached on edge conical shell, which is located under external pressure, it is equal to

$$p=\frac{2E\delta^2\alpha^2}{\sqrt{3}\,(1-\nu^2)\,r^2},$$

where δ - thickness of shell, r - a radius of its basis/base, and α - an angle between plane of basis/base and generatrix.

Formula is derived under the assumption about the axial symmetry of the form of loss of stability.

In conclusion let us examine a question concerning loss of stability and critical load for slightly curved shell of rotation, which is located under the action of the external pressure, evenly

distributed along certain parallel. In the study of this problem principle B it is not possible to directly use. Reason consists of following. If the effective on shell load is continuously distributed over surface, then strain energy is localized in the vicinity of the boundary of bulge, while the load in essence is located out of this vicinity. Specifically, on this basis/base we during the determination of strain energy simply minimized functional U, disregarding the effective on shell load in the vicinity of the boundary of bulge. In now the task in question the load, as strain energy, is concentrated in the vicinity of the boundary of bulge. Therefore, varying the form of shell in the vicinity indicated, it is not possible to disregard the produced by external load work.

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Let for certainty the effective on shell effort/force Q be transferred through the rigid ring, adjacent to shell by the parallel of a radius ρ. In this case, it is logical to assume that the deformation of shell at the moment of loss of stability possesses axial symmetry. The expression of strain energy in standardized variables u, v, which assign deformation, takes the form

$$U = \frac{\pi E \delta^{3/2} \alpha^{5/2} \rho^{*1/2}}{12^{3/4}(1-\nu^2)} \int\limits_{-\infty}^{\infty} (v'^2 + u^2)\, ds.$$

The produced by pressure ring work by the bulge of shell is equal to

$$A = \frac{Q}{12^{1/4}} \sqrt{\delta \rho^* \alpha} \int\limits_{-\infty}^{s^*} v\, ds.$$

In these formulas ρ^{*} - a radius of the circle of bulge, and s^{*} - the value of the dimensionless parameter s, which corresponds to a radius of ring.

From demonstrative considerations it is logical to consider that $\rho^{*}=\rho$. As far as value is concerned s^{*}, about it it is possible to say following. It is obvious, sagging/deflection will be maximum along the parallel where fits closely ring. Therefore s^{*} corresponds to the maximum of the integral, which is determining sagging/deflection. Thus, it is possible to count that

$$\int_{-\infty}^{s^*} v\,ds = \max_{(s)} \int_{-\infty}^{s} v\,ds.$$

The determination of the state of elastic equilibrium with noticeable bulge is reduced to task to the extremum of the functional

$$W = U - A,$$

where the varied functions u, v are connected by the relationship/ratio

$$u' + v + \frac{v^2}{2} = 0 \qquad (**)$$

and they satisfy the boundary conditions

$$u(\infty) = u(-\infty) = 0, \quad v(\infty) = v(-\infty) = 0.$$

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This variational problem is solved by the numerically following method. On the basis of the demonstrative representation of the

character of the examined deformations, we count function $v(s)$ in the vicinity of the parallel where occurs loading, by the function of the form

$$v(s)=\lambda(\sigma-|s|), \quad |s|\leqslant\sigma,$$

where λ and σ - some parameters. In this case, function $u(s)$ in the same interval $|s|\leqslant\sigma$ is determined by communication/connection (**). Out of this interval, i.e., with $|s|\geqslant\sigma$, functions u, v are determined from stability condition of functional U and of condition of the smoothness of coupling with values within interval.

If we by the method indicated find functions $u(s)$, $v(s)$ and to substitute them in the expressions of functionals U and A, then the latter will become the specific functions of the parameters λ and σ. Now, record/fixing the value of function A, i.e., the value of the integral

$$K=\int_{-\infty}^{s^*} v\,ds,$$

let us search for the minimum of function U or, which is the same, minimum of the integral

$$J=\int_{-\infty}^{\infty}(v'^2+u^2)\,ds.$$

Let us designate this minimum J(K). now our variational problem is reduced to task to the extremum of the function

$$W=\frac{\pi E\delta^{5/2}\alpha^{5/2}\rho^{1/2}}{12^{5/4}(1-\nu^2)}J(K)-\frac{Q}{12^{1/4}}\sqrt{\delta\rho\alpha}\,K.$$

From stability condition of this function, we find the received by shell load Q with bulging:

$$Q=\frac{\pi E\delta^2\alpha^2}{\sqrt{12}(1-\nu^2)}\frac{dJ}{dK}.$$

Sudo # 3

Upper critical pressure answers maximum dJ/dK. Numerical calculation according to the described diagram gives for this value of dJ/dK a value ≃3.

Consequently, critical force Q_e which effects on slightly curved shell of rotation through the rigid ring, is determined from the formula

$$Q_e=\frac{3\pi E\delta^2\alpha^2}{\sqrt{12}\,(1-\nu^2)}.$$

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Let us recall that here ρ - radius of the parallel on which fits closely ring to shell, α - an angle between the plane of parallel and the tangential planes of surface, δ - thickness of shell, and E and ν - the elastic constants of material. For the spherical shell of radius R $\alpha \simeq \rho/R$ and, therefore,

$$Q_e=\frac{3\pi E\delta^2\rho^2}{\sqrt{12}\,(1-\nu^2)\,R^2}.$$

Formula for critical force Q_e in the case of spherical shell was subjected to experimental check. Tested spherical shell 1 freely rested on rigid ring 2 (Fig. 23). The action of load Q through vertical rod 3 was transferred to cap/hood by 4, which rested on shell along the parallel of diameter d=2ρ.

Experiment consisted in the fact that load Q, consisting of steel washers 5, gradually increased to the torque/moment of the loss of stability of shell, which was accompanied by distinct click and sharp dropping of rod 3. Value Q, which corresponds to the torque/moment of the loss of stability of shell, was accepted for upper critical value.

For one and the same of shell the experiment was carried out four times with four different cap/hoods by 4, diameter d=8, 10, 12 and 14 mm. Just as in the preceding/previous experiments, the specimen/samples of spherical shells were obtained frcm copper by metal spraying in vacuum to steel form.

In Fig. 24 they are represented graphs of the dependence of upper critical load Q_c on diameter $d=2\rho$ of the parallel, along which is distributed the pressure. Three curves answer three different shells with one and the same with a radius of curvature of R=150 mm, but different thicknesses δ=0.047, δ=0.055, δ=0.072 mm.

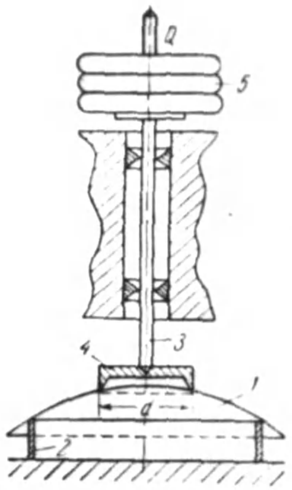

Fig. 23.

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The isolated points, depicted by circles, give the experimental values of upper critical load Q_e, which corresponds to four different values d=8, 10, 12 and 14 mm. As we see, experimental values practical do not differ from theoretical ones.

6. Loss of stability of three-layered shells. The obtained by us results about the stability of flat strictly convex hulls with the different methods of loading can be postponed by the case of the so-called three-layered shells. Three-layered shell consists of the thin skins, prepared from material with high mechanical characteristics, and a comparatively thick layer of filler, prepared from weak material. Just as for usual shells, with the bulge of

three-layered shell energy of its deformation is concentrated in essence on the boundary of the region of bulge and it consists of the strain energy of skins and strain energy of filler.

Designating through u and v shifts (during deformation) of the points of the surface of shell in tangential plane and along the normal respectively, we obtained for the strain energy of skins per the unit of the length γ of the boundary of bulge the following expression:

$$\bar{U}_e = \frac{\delta E}{1-\nu^2} \int_{-\varepsilon^*}^{\varepsilon^*} \left(\frac{\delta^2 v''^2}{12} + \frac{u^2}{\rho^2} \right) ds.$$

Here δ - thickness of skins, ρ - radius of curvature γ, E - modulus of elasticity, ν - Poisson ratio. Integration is fulfilled on the vicinity of the conditional boundary of bulge γ.

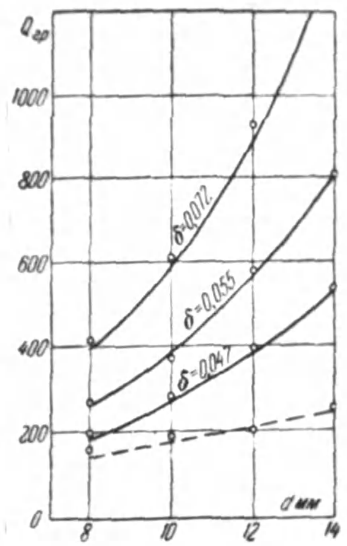

Fig. 24.

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In order to obtain the strain energy of middle layer (filler), we let us assume that the skins are deformed equally (Fig. 25). In this case, in middle layer, are obtained the shearing strains, determined of derivative v, and strain energy per unit volume of filler will be equal to

$$\frac{Gv'^2}{2},$$

where G - modulus of shear of filler. Respectively the strain energy of filler per the unit of the length of the boundary of bulge will be

$$\bar{U}_i = \frac{Gt}{2}\int_{-\rho^*}^{\rho^*} v'^2\, ds,$$

where t - thickness of filler.

Thus, total energy of unit deformation γ of three-layered shell is equal to

$$\bar{U}_e + \bar{U}_i = \frac{\delta E}{1-\nu^2} \int_{-\varepsilon^*}^{\varepsilon^*} \left(\frac{\delta^2 v''^2}{12} + \frac{u^2}{\rho^2} \right) ds + \frac{Gt}{2} \int_{-\varepsilon^*}^{\varepsilon^*} v'^2 \, ds$$

Further, as for usual shells in §1, we standardize the variables u, v, s, set/assuming

$$\bar{u} = \frac{u}{\varepsilon\rho\alpha^2}, \quad \bar{v} = \frac{v'}{\alpha}, \quad \bar{s} = \frac{s}{\rho\varepsilon},$$

where α - an angle between the osculating plane curved γ and the tangential planes of surface, but

$$\varepsilon^4 = \frac{\delta^2}{12\rho^2\alpha^2}.$$

In this case, for strain energy, is obtained the expression

$$\bar{U}_e + \bar{U}_i = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\nu^2)} \int_{-\bar{\varepsilon}^*}^{\bar{\varepsilon}^*} (\bar{v}'^2 + \bar{u}^2)\, d\bar{s} + \frac{Gt\alpha^2\rho\varepsilon}{2} \int_{-\bar{\varepsilon}^*}^{\bar{\varepsilon}^*} \bar{v}^2 \, d\bar{s}.$$

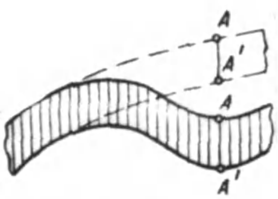

Fig. 25.

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Being limited to the case of such shells and their deformations when the parameter $\delta/\rho\alpha$ is low, let us replace integration limits on $\pm\infty$. Then we obtain

$$\bar{U}_e + \bar{U}_i = \frac{E\delta^{5/2}\alpha^{5/2}\rho^{-1/2}}{12^{3/4}(1-\nu^2)} \int_{-\infty}^{\infty} (v'^2 + u^2)\, ds + \frac{Gt\alpha^2\rho\varepsilon}{2} \int_{-\infty}^{\infty} v^2 \, ds.$$

The feature above the designations of the variables $\bar{u}$, $\bar{v}$, $\bar{s}$ for simplicity of recording here and subsequently lowers.

The form of shell with bulge in the zone of powerful local bending we will determine from the condition of the minimum of energy during the assigned/prescribed common/general/total deformation

$$h=-\frac{1}{2\cdot 12^{1/4}}\sqrt{\delta\rho\alpha}\int_{-\infty}^{\infty}v^2\,ds$$

(see §1). Thus, task is reduced to minimize the functional

$$\bar{U}_e+\bar{U}_i$$

under the condition

$$h=\text{const.}$$

We have

$$\frac{Gt\alpha^2\rho\varepsilon}{2}\int_{-\infty}^{\infty}v^2\,ds=tG\alpha h.$$

Consequently, functional U for a three-layered shell differs from the appropriate functional for a usual shell in terms of term/component/addend, not depending on the varied functions u, v. This means that now the functional in question obtains the steady-state value with the same u, v, as in the appropriate task from §1.

Utilizing that obtained into §1 result, we find for the strain energy of three-layered shell the expression

$$\bar{U}_e+\bar{U}_i=\frac{2E\delta^2\alpha^2h}{\sqrt{12}\,(1-\nu^2)\rho}+tG\alpha h.$$

In order to obtain total energy of the deformation of shell, this expression it is necessary to integrate over arc curved γ, that

limits the region of bulge.

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The integration of first term in expression $\bar{U}_e + \bar{U}_i$ is carried out in §1 chapter 1. If we use the obtained there result, then we will obtain

$$\int_{\gamma} \bar{U}_e \, ds_{\gamma} = 2 \, \frac{4\pi E\delta^2 h\lambda^2}{\sqrt{12}\,(1-\nu^2)\sqrt{R_1 R_2}}\,,$$

where R_1 and R_2 - main radii of curvature in the center of bulge.

Let us find now

$$\int_{\gamma} \bar{U}_i \, ds_{\gamma}.$$

We have (§1 chapter 1)

$$\alpha = \rho k_n,$$

where k_n - normal surface curvature of shell in direction γ. Since

$$\frac{1}{\rho} = \frac{\sqrt{R_1 R_2}}{\lambda\,(R_1 \sin^2\varphi + R_2 \cos^2\varphi)^{3/2}}\,,$$

$$k_n = \frac{1}{R_1 \sin^2\varphi + R_2 \cos^2\varphi}\,,$$

$$ds_{\gamma} = \lambda\,(R_1 \sin^2\varphi + R_2 \cos^2\varphi)^{1/2}\, d\varphi,$$

$$\alpha\, ds_{\gamma} = \frac{\lambda^2\,(R_1 \sin^2\varphi + R_2 \cos^2\varphi)}{\sqrt{R_1 R_2}}\, d\varphi,$$

that hence we have

$$\int_{\gamma} \bar{U}_i \, ds_{\gamma} = Ght \int_{\gamma} \alpha\, ds_{\gamma} = \frac{\pi Gt\,(R_1 + R_2)}{\sqrt{R_1 R_2}}\, h\lambda^2.$$

Thus, total energy of the deformation of shell is equal to

$$U = \frac{8\pi E\delta^2 h\lambda^2}{\sqrt{12}\,(1-\nu^2)\sqrt{R_1 R_2}} + \frac{\pi Gt\,(R_1 + R_2)}{\sqrt{R_1 R_2}}\, h\lambda^2.$$

Produced by the external pressure p work

$$A=\pi p\sqrt{R_1R_2}\,h\lambda^2.$$

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Now from the condition of the equilibrium of the shell

$$d(U-A)=0,$$

where is varied parameter $h\lambda^2$., we find value of the received by shell pressure with bulge, i.e., the critical pressure

$$p=\frac{4E\delta^2}{\sqrt{3}\,(1-\nu^2)\,R_1R_2}+\frac{Gt\,(R_1+R_2)}{R_1R_2}.$$

Let us recall that here R_1 and R_2 - main radii of curvature of shell, δ - the thickness of skins, t - the thickness of the layer of filler, E and ν - the module/modulus of elasticity and Poisson ratio for skins, G - shear modulus for a filler.

Formula for p can be presented even in this form:

$$p=\frac{4E\delta^2K}{\sqrt{3}\,(1-\nu^2)}+2GtH,$$

where K - Gaussian, and H - mean curvature of shell in the center of bulge.

§2. Special infinitesimal bending of strictly convex surface.

The application/use of principle B for the study of the loss of stability of shells and determination of critical loads assumes the construction infinitesimal bending of the surface of shell with

discontinuous change on the boundary of the predicted region of bulge. In the general case for arbitrary surface and arbitrary boundary of bulge, this task is sufficiently difficult. However, in row/series practical of the important tasks of the region of bulging are small and have a form, close to ellipse. In these cases for shells with sufficient regularity of surface, the task of the construction infinitesimal bending indicated can be solved in the closed form. This solution will be given in present paragraph.

Let us recall the formulation of the problem. F - regular strictly convex surface and G - low elliptical region on it with center P. It is required to construct the field infinitesimal bending τ, regular everywhere, besides the boundary γ of region G where it is disruptive, this breakage satisfying the following condition of coupling:

$$\Delta\tau = \sigma e,$$

where e - the unit vector of binormal curved γ, and σ - certain function, assign/prescribed lengthwise γ. In connection with concrete/specific/actual application/appendices we still will assume that the field τ disappears (it vanishes) during removal from the region of bulge.

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f

1. General idea for bending fields. P - center of the region of extrusion G. Since the essential deformations of surface F are limited to small vicinity of point P, logical to introduce the rectangular Cartesian coordinates x, y, z, after accepting tangential plane at point P for plane xy, standard of surface for Z-axis, and point itself P in the origin of coordinates. In this case, if we for the direction of axle/axes x, y accept main directions at point P, then surface F near this point can be assigned by the equation

$$z=\frac{1}{2}(ax^2+by^2),$$

where a and b - the principal curvatures of surface at point P. In the simplest case when region G is coaxial with the indicatrix of curvature at point P, it is assigned by the inequality

$$Ax^2+By^2\leqslant 1.$$

Let us introduce on the surface of coordinate u, v, set/assuming

$$u=x\sqrt{a},\quad v=y\sqrt{b}.$$

In these coordinates our surface is assigned by the equations

$$x=\frac{u}{\sqrt{a}},\quad y=\frac{v}{\sqrt{b}},\quad z=\frac{1}{2}(u^2+v^2).$$

ξ, η, ζ - components of the bending field along the axes x, y, z respectively. From the equation infinitesimal bending

$$dr\,d\tau=0,$$

where r - vector of the point of surface, and τ - vector of the bending field, is obtained the following system for functions ξ, η, ζ:

$$\frac{1}{\sqrt{a}}\xi_u+u\zeta_u=0,$$
$$\frac{1}{\sqrt{b}}\eta_v+v\zeta_v=0,$$
$$\frac{1}{\sqrt{b}}\eta_u+\frac{1}{\sqrt{a}}\xi_v+u\zeta_v+v\zeta_u=0.$$

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If we from this system exclude functions ξ and η, then for ζ is obtained the equation of Laplace

$$\frac{\partial^2\zeta}{\partial u^2}+\frac{\partial^2\zeta}{\partial v^2}=0.$$

Set/assuming

$$w=u+iv,$$

we can present the common/general/total expression for component ζ with the aid of analytic complex variable function w

$$\zeta=\operatorname{Re}\zeta(w).$$

As far as two other components of bending field ξ and η, are concerned, they through the function $\zeta(w)$ are expressed on the formulas

$$\xi=\sqrt{a}\,\operatorname{Re}\left(-u\zeta(w)+\int\zeta(w)\,dw\right),$$

$$\eta=\sqrt{b}\,\operatorname{Re}\left(-v\zeta(w)-i\int\zeta(w)\,dw\right).$$

The obtained representation for the bending fields in equal degrees is related also to the general case. However, keeping in mind the solution of the task of coupling, it is represented by advisable

to expression for the which bends field in the general case to give somewhat other form.

The region of extrusion G in the general case is assigned by the inequality

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 \leqslant 1.$$

Let us introduce the new variables u, v according to the formulas

$$x = \lambda_{11}u + \lambda_{12}v,$$
$$y = \lambda_{21}u \;-,\; {}_{2}v.$$

Coefficients λ_{ij} of these formulas let us select so that in the coordinates u, [redacted] v surface would be assigned by the equation

$$z = \frac{1}{2}(u^2 + v^2),$$

while the region of extrusion G by the inequality

$$Au^2 + Bv^2 \leqslant 1.$$

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The possibility to select coefficients λ_{ij} thus is guaranteed by the positive certainty of the quadratic forms

$$\frac{1}{2}(ax^2 + by^2), \quad a_{11}x^2 + 2a_{12}xy + a_{22}y^2,$$

which by the transformation indicated are led simultaneously to canonical form.

Let us introduce, furthermore, the variables $\tilde{x}$, $\tilde{y}$ according to the equalities

$$\tilde{x} = x\sqrt{a}, \quad \tilde{y} = y\sqrt{b}.$$

This transformation also leads form of z to the sum of the squares

$$z = \frac{1}{2}(\tilde{x}^2 + \tilde{y}^2).$$

It is obvious, the transformation of the variables $\tilde{x}$, $\tilde{y}$ into u, V is orthogonal and is assigned by the formulas

$$\tilde{x} = u\cos\vartheta - v\sin\vartheta,$$

$$\tilde{y} = u\sin\vartheta + v\cos\vartheta.$$

Consequently, communication/connection between the variables $\tilde{x}$, $\tilde{y}$ and u, v is establish/installed by the formulas of the form

$$x = \frac{1}{\sqrt{a}}(u\cos\vartheta - v\sin\vartheta),$$

$$y = \frac{1}{\sqrt{b}}(u\sin\vartheta + v\cos\vartheta).$$

Angle ϑ is found from that condition that our transformation shifts the form

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 \quad \text{в} \quad Au^2 + Bv^2.$$

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Values A and B are the eigenvalues of the form

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

relative to the form

$$ax^2 + by^2,$$

therefore, are the roots of the characteristic equation

$$\begin{vmatrix} a_{11} - \lambda a & a_{12} \\ a_{21} & a_{22} - \lambda b \end{vmatrix} = 0.$$

As shown above, into variable $\tilde{x}$, $\tilde{y}$ bending field is assigned by the equalities

$$\zeta = \operatorname{Re} \tilde{\zeta}(\tilde{z}),$$
$$\xi = \sqrt{a} \operatorname{Re}\left(-\tilde{x}\tilde{\zeta} + \int \tilde{\zeta}\, d\tilde{z}\right),$$
$$\eta = \sqrt{b} \operatorname{Re}\left(-\tilde{y}\tilde{\zeta} - i \int \tilde{\zeta}\, d\tilde{z}\right),$$

where $\tilde{\zeta}(\tilde{z})$ - analytic complex variable function $\tilde{z}=\tilde{x}+i\tilde{y}$.

Let us pass in these formulas to to the variables u, v. Noting that

$$\tilde{z} = we^{i\vartheta}, \quad w = u + iv,$$

let us have

$$\zeta = \operatorname{Re} \zeta(w),$$
$$\xi = \sqrt{a} \operatorname{Re}\left\{-(u\cos\vartheta - v\sin\vartheta)\zeta + e^{i\vartheta} \int \zeta\, dw\right\},$$
$$\eta = \sqrt{b} \operatorname{Re}\left\{-(u\sin\vartheta + v\cos\vartheta)\zeta - ie^{i\vartheta} \int \zeta\, dw\right\},$$

where $\zeta(w) = \tilde{\zeta}(we^{i\vartheta})$ - analytic complex variable function w.

Is such representation for the bending surface fields in the general case.

2. Coupling of bending fields r and r' in simplest case. In p. 1 (page 134) we found general idea for the bending fields with the aid of analytic complex variable function w=u+iv. Specifically,,

$$\zeta = \operatorname{Re} \zeta(w),$$
$$\xi = \sqrt{a} \operatorname{Re}\left(-u\zeta(w) + \int \zeta(w)\, dw\right),$$
$$\eta = \sqrt{b} \operatorname{Re}\left(-v\zeta(w) - i \int \zeta(w)\, dw\right).$$

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To the bending field τ' corresponds analytic function $\zeta'(w)$ within region G, i.e., within the ellipse

$$\frac{A}{a}u^2+\frac{B}{b}v^2=1,$$

while to the bending field τ corresponds analytic function $\zeta(w)$ out of the ellipse indicated.

The which interests us difference in the bending fields to curve γ, i.e., with

$$\frac{A}{a}u^2+\frac{B}{b}v^2=1,$$

is assigned by formula system:

$$\Delta\zeta=\operatorname{Re}\Delta\zeta(w),$$
$$\Delta\xi=\sqrt{a}\operatorname{Re}\left(-u\,\Delta\zeta+\int\Delta\zeta(w)\,dw\right),$$
$$\Delta\eta=\sqrt{b}\operatorname{Re}\left(-v\,\Delta\zeta-i\int\Delta\zeta(w)\,dw\right),$$

where $\Delta\zeta(w)$ - designates a difference analytic functions $\zeta(w)$ and $\zeta'(w)$ on the ellipse

$$\frac{A}{a}u^2+\frac{B}{b}v^2=1.$$

Let us pass from complex variable w to variable ω, set/assuming

$$w=\lambda\omega+\frac{\mu}{\omega}.$$

Let us select constants λ and μ in such a way that to the circumference $|\omega|=1$ from the plane of complex variable ω would correspond the ellipse

$$\frac{A}{a}u^2+\frac{B}{b}v^2=1$$

on plane w. It is obvious, for this it suffices to subordinate values λ, μ to the conditions

$$\lambda+\mu=\sqrt{\frac{a}{A}}, \quad \lambda-\mu=\sqrt{\frac{b}{B}}.$$

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On boundary of the region G, i.e., on the ellipse

$$\frac{A}{a}u^2+\frac{B}{b}v^2=1,$$

$\omega=e^{i\varphi}$ and, therefore,

$$u=(\lambda+\mu)\cos\varphi, \quad v=(\lambda-\mu)\sin\varphi.$$

Let us pass in the formulas, which assign $\Delta r=r-r'$, from the variable w to $\omega=e^{i\varphi}$, set/assuming

$$\Delta\zeta(w)=P(\varphi)+iQ(\varphi).$$

Then we are have

$$\Delta\zeta=P,$$

$$\Delta\xi=-\sqrt{a}\Big\{(\lambda+\mu)P\cos\varphi+ \\ +\int((\lambda+\mu)P\sin\varphi+(\lambda-\mu)Q\cos\varphi)\,d\varphi\Big\},$$

$$\Delta\eta=\sqrt{b}\Big\{-(\lambda-\mu)P\sin\varphi+ \\ +\int((\lambda-\mu)P\cos\varphi-(\lambda+\mu)Q\sin\varphi)\,d\varphi\Big\}.$$

Let us compose equation by curve γ - boundary of the region G on surface, after accepting as the parameter on this curved angle φ=arg ω. We have

$$x=\frac{u}{\sqrt{a}}=\frac{\lambda+\mu}{\sqrt{a}}\cos\varphi,$$

$$y=\frac{v}{\sqrt{b}}=\frac{\lambda-\mu}{\sqrt{b}}\sin\varphi,$$

$$z=\frac{u^2+v^2}{2}=\frac{\lambda^2+\mu^2}{2}+\lambda\mu\cos 2\varphi.$$

Thus, our curve is assigned by the equations

$$x = \frac{\lambda + \mu}{\sqrt{a}} \cos \varphi, \quad y = \frac{\lambda - \mu}{\sqrt{b}} \sin \varphi,$$
$$z = \frac{\lambda^2 + \mu^2}{2} + \lambda\mu \cos 2\varphi.$$

Let us find the vector of binormal by curve γ. Its components along the axes x, y, z are the minors of the matrix/die

$$\begin{pmatrix} x' & y' & z' \\ x'' & y'' & z'' \end{pmatrix}.$$

Page 139. Let us assume

$$a_1 = \begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix}, \quad a_2 = \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix}, \quad a_3 = \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}.$$

Lowering the appropriate unpacking/facings, let us give the resultant expression for values a_1, a_2, a_3:

$$a_1 = -\frac{4}{\sqrt{b}} (\lambda - \mu) \lambda\mu \cos^3 \varphi,$$
$$a_2 = \frac{4}{\sqrt{a}} (\lambda + \mu) \lambda\mu \sin^3 \varphi,$$
$$a_3 = \frac{\lambda^2 - \mu^2}{\sqrt{ab}}.$$

The condition of the coupling of the bending fields it is possible to write in the form

$$\Delta\xi = \sigma a_1,$$
$$\Delta\eta = \sigma a_2,$$
$$\Delta\zeta = \sigma a_3,$$

after include/connecting the factor, which standardizes the vector of binormal, in σ.

Let us differentiate these conditions of coupling for φ and will

substitute in them the obtained above expressions for $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$. We have:

$$(\Delta\zeta)' = P',$$
$$(\Delta\xi)' = -\sqrt{a}\,\{(\lambda+\mu)P'\cos\varphi + Q(\lambda-\mu)\cos\varphi\},$$
$$(\Delta\eta)' = -\sqrt{b}\,\{(\lambda-\mu)P'\sin\varphi + Q(\lambda+\mu)\sin\varphi\}.$$

The conditions of coupling take the form

$$(\lambda+\mu)P'\cos\varphi + Q(\lambda-\mu)\cos\varphi = \frac{4}{\sqrt{ab}}(\lambda-\mu)\lambda\mu(\sigma\cos^3\varphi)',$$
$$(\lambda-\mu)P'\sin\varphi + Q(\lambda+\mu)\sin\varphi = -\frac{4}{\sqrt{ab}}(\lambda+\mu)\lambda\mu(\sigma\sin^3\varphi)',$$
$$P' = \frac{1}{\sqrt{ab}}(\lambda^2-\mu^2)\sigma'.$$

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Including factor $1/\sqrt{ab}$ in σ, we will obtain

$$(\lambda+\mu)P'\cos\varphi + Q(\lambda-\mu)\cos\varphi = 4(\lambda-\mu)\lambda\mu(\sigma\cos^3\varphi)',$$
$$(\lambda-\mu)P'\sin\varphi + Q(\lambda+\mu)\sin\varphi = -4(\lambda+\mu)\lambda\mu(\sigma\sin^3\varphi)',$$
$$P' = (\lambda^2-\mu^2)\sigma'.$$

At first glance under these conditions it is possible to perceive certain incorrectness of our initial task of bending, since function σ, in view of arbitrariness curved γ, is actually arbitrary function, and for two functions P and Q are obtained three equations. However, as can easily be seen, the third equation is the consequence of first two. Thus, we have in the essence of two equations for two functions P and Q, from which they are determined.

In the which interests us application/appendix where for the region of extrusion G is accepted in known sense the simplest form -

ellipse, it is advisable to take as σ the simplest function, such as is constant[1].

FOOTNOTE [1]. There is the foundations for expecting that both our assumptions - that the low region of bulge has a form of ellipse and that the function σ=const - they escape/ensue from the solution of the common/general/total variational problem to which is reduced the problem. However, for this, would be required a more detailed investigation of a question. ENDFOOTNOTE.

Entire further examination will be conducted for this case.

Thus,

$$\sigma = \text{const}.$$

Then

$$P = (\lambda^2 - \mu^2)\sigma, \quad Q = -6\lambda\mu\sigma \sin 2\varphi.$$

Consequently, for analytic functions ζ and ζ', that assign our bending fields, to curve γ is satisfied the condition

$$\zeta - \zeta' = (\lambda^2 - \mu^2)\sigma - i6\lambda\mu\sigma \sin 2\varphi. \qquad (*)$$

Let us find now analytic functions themselves ζ and ζ'. In this case, from function ζ, we will require so that determined by it which bends the field τ out of region G it would disappear at infinity. This condition is dictated by the subsequent application/appendices. It will be carried out, if we require so that at infinity function ζ

would decrease as $1/\omega^2$.

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Condition (*) determines riemann's known task, which in our case has unique solution. Let us find this solution. Let us assume

$$\zeta = \frac{\alpha}{\omega^2}, \quad \zeta' = \beta w^2 + c,$$

where α, β and c - some constants.

On boundary of the region G, i.e., with $|\omega|=1$, will be

$$\zeta = \frac{\alpha}{\omega^2} = \alpha(\cos 2\varphi - i\sin 2\varphi),$$

$$\zeta' = \beta\left(\lambda\omega + \frac{\mu}{\omega}\right)^2 + c =$$
$$= \beta\{(\lambda^2+\mu^2)\cos 2\varphi + i(\lambda^2-\mu^2)\sin 2\varphi + 2\lambda\mu\} + c.$$

Hence

$$\zeta - \zeta' = [\alpha - \beta(\lambda^2+\mu^2)]\cos 2\varphi -$$
$$- i[\alpha + \beta(\lambda^2-\mu^2)]\sin 2\varphi - 2\lambda\mu\beta - c.$$

Taking now into consideration condition (1), we obtain for constants α, β c the following system of equations:

$$\alpha - \beta(\lambda^2+\mu^2) = 0,$$
$$\alpha + \beta(\lambda^2-\mu^2) = 6\lambda\mu\sigma,$$
$$-2\lambda\mu\beta - c = (\lambda^2-\mu^2)\sigma.$$

From this system we find

$$\beta = \frac{3\mu\sigma}{\lambda}, \quad \alpha = \frac{3\mu\sigma}{\lambda}(\lambda^2+\mu^2).$$

After determining functions ζ and ζ', we found thereby the bending fields τ, τ' and solved the task of the bending of surface F,

placed in p. 1. If initial surface is assigned by vector function r, then the isometrically converted surface is assigned by vector function r+r' within the region of extrusion and vector function r+r out of this region. Vector functions r and r' are determined with the aid of analytic functions ζ and ζ' from the formulas, obtained in this p. 2.

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3. Coupling of bending fields in the general case. Just as in the examined simplest case, the bending fields τ and τ' on the boundary γ of the region of extrusion satisfy the condition of the coupling

$$\tau - \tau' = \sigma e,$$

where e - a vector of binormal curved γ, and σ - certain function, assign/prescribed in this curve.

for the bending fields τ and τ', occurs the representation with the aid of the appropriate analytic functions ζ(w):

$$\zeta = \operatorname{Re} \zeta(w),$$
$$\xi = \sqrt{a} \operatorname{Re}\left\{-(u\cos\vartheta - v\sin\vartheta)\zeta(w) + e^{i\vartheta}\int \zeta(w)\,dw\right\},$$
$$\eta = \sqrt{b} \operatorname{Re}\left\{-(u\sin\vartheta + v\cos\vartheta)\zeta(w) - ie^{i\vartheta}\int \zeta(w)\,dw\right\}.$$

For a field τ, the function ζ(w) is analytical in the region

$$Au^2 + Bv^2 \geqslant 1.$$

while for a field r', the corresponding function $\zeta'(w)$ is analytical in the region

$$Au^2 + Bv^2 \leqslant 1.$$

The boundary of these regions is the ellipse, given by the equation

$$Au^2 + Bv^2 = 1.$$

A difference in the bending fields $r-r'$ by lengthwise curve γ is assigned by the system of the formulas

$$\Delta\zeta = \operatorname{Re} \Delta\zeta(w).$$

$$\Delta\xi = \sqrt{a} \operatorname{Re} \left\{ -(u \cos\vartheta - v \sin\vartheta)\Delta\zeta + e^{i\vartheta} \int \Delta\zeta(w)\, dw \right\},$$

$$\Delta\eta = \sqrt{b} \operatorname{Re} \left\{ -(u \sin\vartheta + v \cos\vartheta)\Delta\zeta - ie^{i\vartheta} \int \Delta\zeta(w)\, dw \right\},$$

where $\Delta\zeta(w)$ is a difference analytic functions $\zeta(w)$ and $\zeta'(w)$ on the ellipse

$$Au^2 + Bv^2 = 1.$$

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Just as in the simplest case, we introduce complex variable ω, set/assuming

$$w = \lambda\omega + \frac{\mu}{\omega},$$

and constants λ, μ we determine from that condition so that to the unit circle $|\omega|=1$ on plane w would correspond the ellipse

$$Au^2 + Bv^2 = 1.$$

For this, it is necessary to require, in order to

$$\lambda + \mu = \frac{1}{\sqrt{A}}, \quad \lambda - \mu = \frac{1}{\sqrt{B}}.$$

On boundary of the region G, i.e., to curve γ,

$$\omega = e^{i\varphi}$$

and, therefore,

$$u = (\lambda + \mu)\cos\varphi, \quad v = (\lambda - \mu)\sin\varphi.$$

Let us pass in the formulas, which assign a difference in the bending fields $\Delta\tau$ by lengthwise curve γ, from the variable w to $\omega = e^{i\varphi}$ Set/assuming, as in the simplest case on γ

$$\Delta\zeta(w) = P(\varphi) + iQ(\varphi),$$

let us have:

$$\begin{aligned}\Delta\xi = \sqrt{a}\,\mathrm{Re}\Big\{&-(\lambda\cos(\varphi+\vartheta) + \mu\cos(\vartheta-\varphi))(P+iQ) + \\ &+ ie^{i\vartheta}\int (P+iQ)(\lambda e^{i\varphi} - \mu e^{-i\varphi})\,d\varphi\Big\} = \\ = \sqrt{a}\,\Big\{&-(\lambda\cos(\varphi+\vartheta) + \mu\cos(\vartheta-\varphi))P - \\ &- \int(\lambda\sin(\varphi+\vartheta) - \mu\sin(\vartheta-\varphi))P\,d\varphi - \\ &- \int(\lambda\cos(\varphi+\vartheta) - \mu\cos(\vartheta-\varphi))Q\,d\varphi\Big\}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{d\varphi}(\Delta\xi) = -\sqrt{a}\,\{&(\lambda\cos(\varphi+\vartheta) + \mu\cos(\vartheta-\varphi))P' + \\ &+ (\lambda\cos(\varphi+\vartheta) - \mu\cos(\vartheta-\varphi))Q\}.\end{aligned}$$

Analogously it is obtained

$$\begin{aligned}\frac{d}{d\varphi}(\Delta\eta) = -\sqrt{b}\,\{&(\lambda\sin(\vartheta+\varphi) + \mu\sin(\vartheta-\varphi))P' + \\ &+ (\lambda\sin(\varphi+\vartheta) - \mu\sin(\vartheta-\varphi))Q\}.\end{aligned}$$

Finally,

$$\frac{d}{d\varphi}(\Delta\zeta) = P'.$$

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If we designate the components of the vector of the binormal of curve through a_1, a_2, a_3, then the conditions of coupling after differentiation with respect to ϕ will be written as follows:

$$\frac{d}{d\varphi}(\Delta\xi) = (a_1\sigma)',$$
$$\frac{d}{d\varphi}(\Delta\eta) = (a_2\sigma)',$$
$$\frac{d}{d\varphi}(\Delta\zeta) = (a_2\sigma)'.$$

Let us find expressions for components a_1, a_2, a_3 and will substitute them into these formulas.

Surface F in coordinates u, v is assigned by the equations

$$x = \frac{1}{\sqrt{a}}(u\cos\vartheta - v\sin\vartheta),$$
$$y = \frac{1}{\sqrt{b}}(u\sin\vartheta + v\cos\vartheta),$$
$$z = \frac{1}{2}(u^2 + v^2).$$

By lengthwise curve γ

$$u = (\lambda + \mu)\cos\varphi, \quad v = (\lambda - \mu)\sin\varphi.$$

Substituting these values of u, v into the equations of surface, we will obtain equations by curve γ:

$$x = \frac{1}{\sqrt{a}}((\lambda + \mu)\cos\varphi\cos\vartheta - (\lambda - \mu)\sin\varphi\sin\vartheta),$$
$$y = \frac{1}{\sqrt{b}}((\lambda + \mu)\cos\varphi\sin\vartheta + (\lambda - \mu)\sin\varphi\cos\vartheta),$$
$$z = \frac{\lambda^2 + \mu^2}{2} + \lambda\mu\cos 2\varphi.$$

With the aid of the equations of curve, we find the components of the vector of the binormal

$$a_1 = \begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix} = -\frac{4\lambda\mu}{\sqrt{b}}(\lambda + \mu)\sin\vartheta\sin^3\varphi - \frac{4\lambda\mu}{\sqrt{b}}(\lambda - \mu)\cos\vartheta\cos^3\varphi,$$
$$a_2 = \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix} = \frac{4\lambda\mu}{\sqrt{a}}(\lambda + \mu)\cos\vartheta\sin^3\varphi - \frac{4\lambda\mu}{\sqrt{a}}(\lambda - \mu)\sin\vartheta\cos^3\varphi,$$
$$a_3 = \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix} = \frac{1}{\sqrt{ab}}(\lambda^2 - \mu^2).$$

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Substituting the obtained values a_1, a_2, a_3 under the conditions of coupling and by including factor $1/\sqrt{ab}$ in σ, we will obtain

$$-(\lambda\cos(\varphi+\vartheta)+\mu\cos(\vartheta-\varphi))P'-Q(\lambda\cos(\varphi+\vartheta)-$$
$$-\mu\cos(\vartheta-\varphi))=-4\lambda\mu\{\sigma(\lambda+\mu)\sin\vartheta\sin^3\varphi+$$
$$+\sigma(\lambda-\mu)\cos\vartheta\cos^3\varphi\}'.$$
$$-(\lambda\sin(\varphi+\vartheta)+\mu\sin(\vartheta-\varphi))P'-Q(\lambda\sin(\varphi+\vartheta)-$$
$$-\mu\sin(\vartheta-\varphi))=4\lambda\mu\{\sigma(\lambda+\mu)\cos\vartheta\sin^3\varphi-$$
$$-\sigma(\lambda-\mu)\sin\vartheta\cos^3\varphi\}'.$$
$$P'=\{\sigma(\lambda^2-\mu^2)\}'.$$

Multiplying the second equality on i and adding to the first, we obtain

$$-P'(\lambda e^{i(\varphi+\vartheta)}+\mu e^{i(\vartheta-\varphi)})-Q(\lambda e^{i(\varphi+\vartheta)}-\mu e^{i(\vartheta-\varphi)})=$$
$$=\{4\lambda\mu\sigma(\lambda+\mu)e^{i\vartheta}i\sin^3\varphi-4\lambda\mu\sigma(\lambda-\mu)e^{i\vartheta}\cos^3\varphi\}'.$$

If now this equality is reduced on $e^{i\vartheta}$ and to separate/liberated real part of apparent/imaginary, then are obtained the following two relationship/ratios:

$$P'(\lambda+\mu)\cos\varphi+Q(\lambda-\mu)\cos\varphi=\{4\lambda\mu(\lambda-\mu)\sigma\cos^3\varphi\}',$$
$$P'(\lambda-\mu)\sin\varphi+Q(\lambda+\mu)\sin\varphi=-\{4\lambda\mu(\lambda+\mu)\sigma\sin^3\varphi\}'.$$

We see that in the general case for functions P and Q is

obtained accurately the same system of equations as in the simplest case. Just as there, we set/assume

$$\sigma = \text{const}.$$

Then

$$P = (\lambda^2 - \mu^2)\sigma,$$
$$Q = -6\lambda\mu\sigma \sin 2\varphi.$$

After this, just as in the examined case, we find analytic functions ζ and ζ', the assigning bending field

$$\zeta = \frac{\alpha}{\omega^2}, \quad \zeta' = \beta w^2 + c,$$
$$\beta = \frac{3\mu}{\lambda}\sigma, \quad \alpha = \frac{3\mu}{\lambda}\sigma(\lambda^2 + \mu^2).$$

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With the aid of functions ζ and ζ^*, we find through the appropriate formulas the bending fields τ and τ^*, but with them and the vector function, which assigns the isometrically converted surface.

3. Loss of stability of the shells of revolution with the different methods of loading.

Among strictly convex hulls most widely used, perhaps, are the shells of revolution. In connection with this the study of the problem concerning the stability of such shells is of considerable practical interest. In present paragraph we will examine the loss of stability of the convex hulls of rotation with the different methods of the loading: internal pressure, external pressure and twisting. In particular, we will determine critical loads in each of these cases.

To the study of the problem concerning the stability of the shells of revolution we will premise the supplementary investigation of a question concerning the loss of stability of flat strictly

convex hulls under uniform external pressure. As shown in §1, the rigidity of the attachment of the edge of this shell predetermines the form of the region of bulge and the character of the bending fields, in which is examined the variational problem for functional W=U-A. Now we will not require the special rigidity of the attachment of edge, but will assume the smallness of the region of bulge and its form let us consider it elliptical.

1. Loss of stability of strictly convex hull under uniform external pressure. Let the loss of stability of shell under external pressure be accompanied by the bulge of finite domain G of the elliptic form of small size/dimensions. It is logical to consider that region G is coaxial with the indicatrix of curvature in its center P. In particular, region G and indicatrix can be similar and similarly arrange/located, which corresponds to the case, examined in §1.

The form of shell with noticeable bulge we will approach the isometric conversion of initial surface, which is obtained in §2 (see end p. 2).

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In this case, as shown in §1, the strain energy of shell is concentrated in essence in curve γ - boundary of the region of bulge,

and per the unit of length by curve γ it is determined from the formula

$$\overline{U} = \frac{2E\delta^{2}\alpha^{2}h}{\sqrt{12(1-\nu^{2})}\,\rho}.$$

Here h - normal sagging/deflection on the region of bulge along boundary γ, ρ - radius of curvature γ, α - angle between the osculating plane curved γ and tangential planes of surface, E - modulus of elasticity, ν - Poisson ratio.

For the isometric conversion of surface, constructed in § 2, along the boundary γ of the region of bulge we have

$$h = \mathrm{Re}\,(\Delta\zeta)_{\gamma},$$

where $(\Delta\zeta)_{\gamma}$ - a difference along γ analytic functions, which assign the bending fields out of region G and within this region.

Since the region of extrusion G is small, then angle α can be determined from the formula

$$\alpha = \frac{k_n}{k},$$

where k - curvature curved γ, and k_n - normal curvature of initial surface in direction γ. Let us calculate k and k_n.

Let us introduce as earlier the system of Cartesian coordinates

x, y, z, after accepting tangential plane in the center of bulge P for plane xy, normal to the surface - for z-axis, and point P - in the origin of coordinates. Of the axes of coordinates x and y is directed along tangents to camber lines at point P. Then near point P surface is assigned by the equation

$$z = \frac{1}{2}(ax^2 + by^2).$$

The first and second quadratic shapes of surface near point P will be

$$\mathrm{I} = dx^2 + dy^2,$$
$$\mathrm{II} = a\,dx^2 + b\,dy^2.$$

Hence for normal surface curvature, is obtained the expression

$$k_n = \frac{a\,dx^2 + b\,dy^2}{dx^2 + dy^2}.$$

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As shown in § 2, curve γ is assigned by the equations

$$x = p\cos\varphi, \quad y = q\sin\varphi,$$

where

$$p = \frac{\lambda+\mu}{\sqrt{a}}, \quad q = \frac{\lambda-\mu}{\sqrt{b}},$$

while λ and μ they are found from the equations

$$\lambda+\mu = \sqrt{\frac{a}{A}}, \quad \lambda-\mu = \sqrt{\frac{b}{B}}.$$

The constants A and B determine the region of bulge G:

$$Ax^2+By^2\leqslant 1.$$

In view of the fact that region G is small, curvature γ can be determined according to its projection on plane xy, which is assigned by the same two equations

$$x=p\cos\varphi,\quad y=q\sin\varphi.$$

In this case, is obtained the following expression for the curvature:

$$k=\frac{pq}{(p^2\sin^2\varphi+q^2\cos^2\varphi)^{3/2}}.$$

Substituting in the common/general/total expression for the normal surface curvature of the expression

$$dx=-p\sin\varphi\,d\varphi,\quad dy=q\cos\varphi\,d\varphi,$$

we will obtain normal surface curvature in direction γ

$$k_n=\frac{(\lambda+\mu)^2\sin^2\varphi+(\lambda-\mu)^2\cos^2\varphi}{p^2\sin^2\varphi+q^2\cos^2\varphi}.$$

If we now substitute the obtained values into formula for $\overline{U}$ and to integrate over arc to curve γ, then we will obtain total energy of strain. We have

$$\frac{\alpha^2}{\rho} = \left(\frac{(\lambda+\mu)^2 \sin^2\varphi + (\lambda-\mu)^2 \cos^2\varphi}{p^2 \sin^2\varphi + q^2 \cos^2\varphi}\right)^2 \frac{(p^2 \sin^2\varphi + q^2 \cos^2\varphi)^{3/2}}{pq},$$

$$ds = (p^2 \sin^2\varphi + q^2 \cos^2\varphi)^{1/2}\, d\varphi.$$

[Page 149.] Hence

$$\int_\gamma \frac{\alpha^2}{\rho}\, ds = \int_0^{2\pi} [(\lambda+\mu)^2 \sin^2\varphi + (\lambda-\mu)^2 \cos^2\varphi]^2 \frac{d\varphi}{pq} = (\lambda^4 + \mu^4 + 4\lambda^2\mu^2)\frac{2\pi}{pq}.$$

Taking into account values of p and q, we will obtain

$$\int_\gamma \frac{\alpha^2}{\rho}\, ds = \frac{2\pi\sqrt{ab}}{\lambda^2-\mu^2}(\lambda^4+\mu^4+4\lambda^2\mu^2).$$

Total energy of strain is equal to

$$U = \int_\gamma \bar{U}\, ds = \frac{2E\,\delta^2 h}{\sqrt{12}\,(1-\nu^2)} \frac{2\pi\sqrt{ab}}{\lambda^2-\mu^2}(\lambda^4+\mu^4+4\lambda^2\mu^2).$$

In view of the smallness of region G, during our selection of the coordinate system value h can be taken as equal to a difference in the component along the axis z bending fields r and r^* (§ 2). In this case,

$$h = P(\varphi) = (\lambda^2-\mu^2)\,\sigma.$$

As a result for the strain energy of shell, is obtained the following resultant expression:

$$U = \frac{2E\,\delta^2\sigma 2\pi\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}(\lambda^4+\mu^4+4\lambda^2\mu^2).$$

Let us calculate that now produced by the external pressure p work A. If a change (during strain) in the volume, limited by shell, is designated ΔV, then the work

$$A = p\,\Delta V.$$

Since region G is small, then essential strains shell experience/tests only near point P. Therefore value ΔV is determined with the aid of the integral

$$\Delta V = \int\int \zeta\, dx\, dy,$$

where ζ - a shift of the points of surface during strain in the direction of z-axis. The value of mixing ζ is determined by two analytic functions $\zeta(w)$ and $\zeta'(w)$.

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Specifically,, out of the region of the bulge

$$\zeta = \mathrm{Re}\,\zeta(w),$$

while within this region

$$\zeta = \mathrm{Re}\,\zeta'(w).$$

(expressions for functions $\zeta(w)$ and $\zeta'(w)$ they are obtained in § 2.)

Thus

$$\Delta V = \mathrm{Re}\left\{\int\int \zeta'(w)\, dx\, dy + \int\int \zeta(w)\, dx\, dy\right\}.$$

Here in first term integration applies to the interior of the ellipse

$$Ax^2 + By^2 = 1,$$

while the second - to the remaining part of plane xy.

Transfer/converting from the variables x, y to to the variables u, v,

$$x = \frac{u}{\sqrt{a}}, \quad y = \frac{v}{\sqrt{b}},$$

let us have

$$\int\int \zeta'(w)\,dx\,dy = \frac{1}{\sqrt{ab}} \int\int \zeta'(w)\,du\,dv,$$

where integration it is fulfilled to the right on the interior of the ellipse

$$\frac{A}{a}u^2 + \frac{B}{b}v^2 = 1$$

on the plane of complex variable w=u+iv. For computing this integral, let us examine the line integral on the boundary of the ellipse

$$J' = \oint \zeta'(w)\,\overline{w}\,dw.$$

Converting integral J' to integral by the area of

ellipse with the aid of grina - Ostrogradskiy's formula and taking into account the analyticity of function $\zeta'(w)$, let us have

$$\oint \zeta'(w)\,\overline{w}\,dw = -2i \int\int \zeta'(w)\,du\,dv.$$

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It is analogous, the integral

$$J = \oint \zeta(w)\,\overline{w}\,dw$$

is converted to integral on the exterior of ellipse. Here it is substantial to note that $\zeta(w)$ decreases at infinity as $1/w^2$. In the same direction of contour integration of ellipse, it will be

$$\oint \zeta(w)\,\overline{w}\,dw = 2i \int\int \zeta(w)\,du\,dv.$$

Substituting the obtained expressions of integrals in formula for ΔV, we will obtain

$$\Delta V = \frac{1}{\sqrt{ab}}\,\mathrm{Re}\,\frac{1}{2i}\oint (\zeta(w) - \zeta'(w))\,\overline{w}\,dw.$$

Let us note that on the outline/contour of the integration

$$\zeta(w) - \zeta'(w) = \Delta\zeta = P + iQ,$$

where P and Q have the following values:

$$P = (\lambda^2 - \mu^2)\,\sigma, \quad Q = -6\lambda\mu\sigma\sin 2\varphi.$$

Let us now move on from the variable w to ω, after assuming

$$w = \lambda\omega + \frac{\mu}{\omega}.$$

On plane w the outline/contour of integration is the unit circle and, therefore, on outline/contour we have

$$w = \lambda e^{i\varphi} + \mu e^{-i\varphi},$$
$$\bar{w} = \lambda e^{-i\varphi} + \mu e^{i\varphi},$$
$$\Delta\zeta = (\lambda^2 - \mu^2)\sigma - 3\lambda\mu\sigma(e^{2i\varphi} - e^{-2i\varphi}).$$

Substituting the obtained expressions w, $\bar{w}$, Δζ in the integral

$$\oint\limits_{|\omega|=1} \Delta\zeta(w)\bar{w}\,dw,$$

we easily fulfill integration and find the following relationship/ratio:

$$\oint\limits_{|\omega|=1} \Delta\zeta(w)\bar{w}\,dw = 2\pi i\sigma(\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

Together with this we obtain

$$\Delta V = \frac{\pi\sigma}{\sqrt{ab}}(\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

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Consequently,

$$A = \frac{\pi p\sigma}{\sqrt{ab}}(\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

Values γ, μ characterize the form of the region of bulge, and σ

- a value of bulge. In connection with this is represented by advisable to introduce the single parameter

$$\varepsilon = \pi\sigma(\lambda^4 + \mu^4 + 4\lambda^2\mu^2),$$

that characterizing the supercritical deformation of shell. Depending on this parameter, the strain energy of shell and the produced by external pressure work are determined from the formulas

$$U = \frac{4E\,\delta^2\sqrt{ab}\,\varepsilon}{\sqrt{12}\,(1-\nu^2)},$$

$$A = \frac{p\varepsilon}{\sqrt{ab}}.$$

Now from the condition of the equilibrium of shell at the moment of the bulge

$$\frac{d}{d\varepsilon}(U - A) = 0$$

we find the received by shell load. We have

$$\frac{4E\,\delta^2\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)} - \frac{p}{\sqrt{ab}} = 0.$$

Hence

$$p = \frac{4E\,\delta^2 ab}{\sqrt{12}\,(1-\nu^2)}.$$

Noting that a also b - principal curvatures of shell in center bulge, we can rewrite this formula in the form

$$p = \frac{2E\,\delta^2}{\sqrt{3}\,(1-\nu^2)\,R_1R_2},$$

where R_1 and R_2 - main radii of curvature.

We see that and under more common/general/total assumptions about the character of the bulge of shell at the moment of loss of stability is obtained the same critical pressure, as in simplest examination of § 1.

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2. Special isometric conversion of convex surface of revolution. Experiment shows that the convex hull of rotation under internal pressure can lose stability with the formation of the system of the correctly arranged/located elliptical dents along certain parallel (Fig. 26). The physical cause for this form of loss of stability lies in the fact that for the dents, elongated in the direction of meridians, during the deformation of shell indicated can occur the common/general/total increase in the limited by it volume, in spite of the extrusion of shell along the system of dents inside this volume [1].

FOOTNOTE [1]. It is possible to give another consideration, which elucidates the loss of stability of the convex hull of rotation under internal pressure. During a specific ratio between the principal curvatures of shell, internal pressure can cause the appearance of

compressive forces in the direction of parallels. These effort/forces at their known intensity cause loss of stability. ENDFOOTNOTE.

Just as in the preceding/previous examination, we will approach the form of shell with bulge the isometric conversion of initial surface. In this case, we will not investigate the parts of this conversion and will be restricted to the determination only of such values, connected with deformation, which by us will be necessary during the solution of a question concerning the stability of shell. In particular, us interests a question concerning that, to what extent they separate from one another of the plane of the parallels, which limit the system of dents.

In view of the fact that the deformation of shell out of the zone of the appearing dents is small, the final bending of surface in this part can be considered infinitesimal bending. The appropriate bending field we will determine via the superposition of the bending fields, connected with the education/formation of separate dents. The bending field, caused by the education/formation of one dent (region of bulge), we take in the same form as it is obtained in § 2.

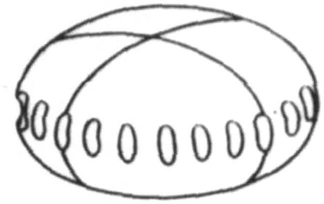

Fig. 26.

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A - arbitrary point of surface, which is located on the small distance of h from the parallel γ, along which are arrange/located the regions of bulge. Let us determine $\tilde{\xi}$ - component of the which bends fields in the direction meridian. P - basis/base of the geodetic perpendicular, omitted from point A to parallel γ. Let us introduce the Cartesian coordinates x, y, z, after accepting tangent to meridian for x axis, tangent to parallel for y axis, and normal to the surface for z-axis and point P in the origin of coordinates.

It is logical to assume that the value of component $\tilde{\xi}$ substantially affect only the bulges of shell near point P. Therefore, if component of the bending field on meridian from the region of bulge with center P is designated $\xi(x, y)$, then which interests us constituting $\tilde{\xi}$, connected with the advent of an entire system of regions of bulge, it will be

$$\tilde{\xi} = \sum_k \xi(h, y_k).$$

where y_k designate the coordinates of the centers of the adjacent regions of bulge.

Let us examine in more detail the function $\xi(x, y)$. In connection with this let us recall that together with by the variables x, y we introduced the variables u, v, connected with by x, y relationship/ratios

$$x\sqrt{a} = u, \quad y\sqrt{b} = v,$$

complex variable

$$w = u + iv$$

and complex variable ω,

$$w = \lambda\omega + \frac{\mu}{\omega}$$

(see § 2). Here a and b - principal curvatures in the center of bulge P, and λ, μ - parameters, which characterize the form of the region of bulge. It is substantial to note that for the low regions of bulge - this by us will be assumed - λ and μ they are small.

In § 2 for component ξ of the bending field, connected with the

advent of one region of bulge, we obtained the following expression:

$$\xi = \sqrt{a}\ \mathrm{Re}\left\{-u\zeta + \int \zeta\, dw\right\}.$$

where $\zeta(w)$ - analytic complex variable function w, which out of the region of bulge is determined from the formula

$$\zeta = \frac{\alpha}{\omega^2}, \quad \alpha = \frac{3\mu}{\lambda}(\lambda^2 + \mu^2)\,\sigma.$$

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Substituting this value ζ in formula for ξ and fulfilling integration, we will obtain

$$\xi = \alpha\sqrt{a}\ \mathrm{Re}\left\{-\frac{u}{\omega^2} - \frac{\lambda}{\omega} + \frac{\mu}{3\omega^3}\right\}.$$

Let us note that at the bounded below value $|w|$ the corresponding value ω, determined by the relationship/ratio

$$w = \lambda\omega + \frac{\mu}{\omega},$$

with small ones λ and μ is sufficiently great in absolute value. Therefore with small ones λ and μ term/component/addend $\mu/3\omega^3$ in formula for ξ can be disregarded; furthermore, it is possible to count that

$$w = \lambda\omega.$$

If we consider these observations, then formula for component ξ can

be presented in the following entreated form:

$$\xi = -\lambda^2\alpha\sqrt{a}\ \mathrm{Re}\left\{\frac{u}{w^2} + \frac{1}{w}\right\}.$$

Isolating real part, we will obtain

$$\xi = -\lambda^2\alpha\sqrt{a}\, u^3 \frac{1}{(u^2+v^2)^2}.$$

Now

$$v_k = \sqrt{b}\, y_k.$$

Then

$$\tilde{\xi} = -\lambda^2\alpha\sqrt{a}\, u^3 \sum_k \frac{1}{(u^2+v_k^2)^2},$$

where

$$u = h\sqrt{a}.$$

Addition in the right side of the equality by hypothesis applies to adjacent regions. However, in view of rapid convergence of series, it is possible to count that the addition is fulfilled on all k, i.e., that

$$\tilde{\xi} = -\lambda^2\alpha\sqrt{a}\, u \sum_{k=-\infty}^{\infty} \frac{1}{(u^2+v_k^2)^2}.$$

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With a sufficient denseness of the placement of the area of bulge, that is the small

$$\Delta v = v_k - v_{k-1},$$

the addition in formula for $\tilde{\xi}$ can be replaced with integration. Then we obtain

$$\tilde{\xi} = -\frac{\lambda^2 \alpha \sqrt{a}}{\Delta v} u^3 \int_{-\infty}^{\infty} \frac{dv}{(u^2 + v^2)^2}.$$

Or, by producing replacement with variable

$$v = ut,$$

let us find

$$\tilde{\xi} = -\frac{\lambda^2 \alpha \sqrt{a}}{\Delta v} \int_{-\infty}^{\infty} \frac{dt}{(1 + t^2)^2}.$$

We have

$$\int_{-\infty}^{\infty} \frac{dt}{(1 + t^2)^2} = \frac{\pi}{2}.$$

Therefore

$$\tilde{\xi} = -\frac{\pi \lambda^2 \alpha \sqrt{a}}{2 \Delta v}.$$

Introducing here the value

$$\alpha = -\frac{3\mu}{\lambda} (\lambda^2 + \mu^2) \sigma,$$

we will obtain the following final formula for $\tilde{\xi}$:

$$\tilde{\xi}=\frac{3\pi\sqrt{a}\,\lambda\mu(\lambda^2+\mu^2)\sigma}{2\Delta v}.$$

If point A is taken from another side of the zone of the regions of bulge, then it during the deformation in question will obtain in accuracy/precision the same shift in meridian, but in the opposite direction.

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Hence it follows that the which interests us removal of the planes of parallels, which limit the zone of the regions of bulge, it occurs to value

$$\varepsilon=\frac{3\pi\sqrt{a}\,\lambda\mu(\lambda^2+\mu^2)\sigma\cos\alpha}{\Delta v},$$

where α - an angle between the tangent to meridian and the axle/axis of surface. If we introduce instead of v variable

$$y=\frac{v}{\sqrt{b}},$$

that

$$\varepsilon=\frac{3\pi}{\Delta y}\sqrt{\frac{a}{b}}\,\lambda\mu(\lambda^2+\mu^2)\sigma.$$

Let us recall that here a and b - normal curvatures of surface in direction meridian and parallel respectively, Δy - distance between

centers of the adjacent regions of bulge, while $\lambda\mu(\lambda^2+\mu^2)\varepsilon$ - value, which characterizes the separately undertaken region of bulge.

By the determination of value ε we will finish our examination of isometrically converted surface and will pass to the investigation of a question concerning the stability of shell.

3. Critical internal pressure for convex hull of rotation. The critical internal pressure, by which occurs the loss of stability of the shell of revolution with the formation of the system of the regions of bulge along certain parallel, we will determine, examining elastic equilibrium with noticeable bulge. The condition of the equilibrium

$$d(U-A)=0,$$

where U - strain energy of shell, A - produced by pressure work.

For energy U_1 of the deformation of shell, connected with the bulge of one region, in p. 1 we found the expression

$$U_1=\frac{2E\,\delta^2 2\pi\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}\,\sigma(\lambda^4+\mu^4+4\lambda^2\mu^2).$$

If we a number of regions of bulge designate n, then the common/general/total strain energy of shell will be

$$U=\frac{2E\,\delta^2 2\pi\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}\,\sigma(\lambda^4+\mu^4+4\lambda^2\mu^2)\,n.$$

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Let us turn now to work A. If the connected with bulge change in the volume, limited by shell, is designated ΔV, then to work it is equal to

$$A = p\,\Delta V.$$

where p - internal pressure.

Let us conduct perpendicularly to the axle/axis of the surface two planes, which limit the zone of the regions of bulge. A change in the volume ΔV is caused by the removal of the planes conducted during the deformation of shell and by the education/formation of very regions of bulge. Let us designate counterparts ΔV through ΔV_e and ΔV_i.

Value ΔV_i is negative and for one region of bulge is determined from formula (st. p. 1)

$$\Delta V_i' = -\frac{\pi\sigma}{\sqrt{ab}}(\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

Consequently, for all n of the regions

$$\Delta V_i = -\frac{\pi\sigma}{\sqrt{ab}}(\lambda^4 + \mu^4 + 4\lambda^2\mu^2)\,n.$$

Further, we have

$$\Delta V_e = \pi\rho^2\varepsilon.$$

where ρ - a radius of the parallel, along which are arrange/located the regions of bulge, and ε - removal of the planes of parallels, which limit the zone of the regions of bulge, during the deformation of shell. Substituting here the value ε, found in p. 2 we will obtain

$$\Delta V_{\epsilon} = \pi\rho^2 \frac{3\pi}{\Delta y} \sqrt{\frac{a}{b}} \lambda\mu (\lambda^2 + \mu^2) \sigma \cos\alpha.$$

Thus, for work A, produced by the internal pressure p, is obtained the expression

$$A = -\frac{\pi\sigma (\lambda^4 + \mu^4 + 4\lambda^2\mu^2) pn}{\sqrt{ab}} + \\ + \pi\rho^2 \frac{3\pi}{\Delta y} \sqrt{\frac{a}{b}} \lambda\mu (\lambda^2 + \mu^2) \sigma p \cos\alpha.$$

Record/fixing the form of the region of bulge (parameters λ, μ), let us vary sagging/deflection in the regions of bulge (parameter σ).

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Then from the condition of the equilibrium

$$\frac{d}{d\sigma}(U - A) = 0$$

is obtained the following relationship/ratio for the value of

pressure p, received by shell with the bulge:

$$\frac{2E\delta^2 2\pi\sqrt{ab}}{\sqrt{12}(1-\nu^2)}(\lambda^4+\mu^4+4\lambda^2\mu^2)+\frac{\pi}{\sqrt{ab}}(\lambda^4+\mu^4+4\lambda^2\mu^2)np-$$
$$-\pi\rho^2\frac{3\pi}{\Delta y}\sqrt{\frac{a}{b}}\lambda\mu(\lambda^2+\mu^2)p\cos\alpha=0.$$

Multiplying this relationship/ratio on

$$\frac{\Delta y}{2\pi^2\rho}\sqrt{ab},$$

and noting that

$$\Delta yn=2\pi\rho,$$

let us have

$$\frac{4E\delta^2 ab}{\sqrt{12}(1-\nu^2)}(\lambda^4+\mu^4+4\lambda^2\mu^2)+$$
$$+(\lambda^4+\mu^4+4\lambda^2\mu^2)p-\frac{3\rho}{2}a\lambda\mu(\lambda^2+\mu^2)p\cos\alpha=0.$$

Let us divide this relationship/ratio on

$$\lambda^4+\mu^4+4\lambda^2\mu^2$$

and let us assume

$$\vartheta=\frac{\lambda\mu}{\lambda^2+\mu^2}.$$

We will obtain

$$\frac{4E\delta^2 ab}{\sqrt{12}(1-\nu^2)}+p-\frac{3a\rho\cos\alpha}{2}\frac{\vartheta p}{1+2\vartheta^2}=0.$$

Hence

$$p = \frac{4E\,\delta^2 ab}{\sqrt{12}\,(1-\nu^2)}\,\frac{1}{\frac{3a\rho\cos\alpha}{2}\,\vartheta^* - 1},$$

where

$$\vartheta^* = \frac{\vartheta}{1+2\vartheta^2}.$$

The parameter ϑ^* by known form is expressed as the parameters λ, μ and, therefore, it characterizes the form of the region of bulge. let us explain the interval of the allowed values of the parameter ϑ^*.

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For this let us, first of all, note that value

$$\vartheta = \frac{\lambda\mu}{\lambda^2+\mu^2}$$

is included between -1/2 and +1/2. Further, ϑ^* monotonically depends on ϑ, since

$$\frac{d\vartheta^*}{d\vartheta} = \frac{1-2\vartheta^2}{(1+2\vartheta^2)^2} > 0.$$

Consequently, ϑ^* is included between -1/3 and +1/3.

Taking into account the interval of the allowed values ϑ^*, we consist that the small pressure at which the shell can lose stability

with bulge along the assigned/prescribed parallel, is determined from the formula

$$p=\frac{2E\,\delta^2ab}{\sqrt{3}\,(1-\nu^2)}\,\frac{1}{\frac{a\rho}{2}\cos\alpha-1}.$$

Let us recall that in this formula a and b - normal surface curvatures along meridian and parallel respectively, ρ - radius of parallel, α - angle between the tangent to meridian and the axle/axis of surface. If we into this formula introduce the main radii of curvature of surface R_1 and R_2,

$$\frac{1}{a}=R_1,\quad \frac{1}{b}=R_2,$$

and to note that

$$\frac{\cos\alpha}{\rho}=\frac{1}{R_2},$$

then our formula takes the form

$$p=\frac{2E\delta^2}{\sqrt{3}\,(1-\nu^2)\,R_1R_2}\,\frac{1}{\frac{\rho^2}{2R_1R_2}-1}.$$

The small value p we was obtained, set/assuming $\vartheta^*=1/3$. To this value ϑ^* corresponds $\vartheta=1/2$. Since

$$\vartheta=\frac{\lambda\mu}{\lambda^2+\mu^2},$$

the this is possible only with $\lambda=\mu$. But this means that the region of bulge, given by the equations

$$x=\frac{\lambda+\mu}{\sqrt{a}}\cos\varphi,\quad y=\frac{\lambda-\mu}{\sqrt{b}}\sin\varphi,$$

degenerates in axis intercept x (meridian).

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The physical sense of this result lies in the fact that the appearing as a result losses of stability of the shell of dent must be strongly elongated along meridians. This really/actually is observed in the appropriate experiments.

Let us use the obtained by us formula to the determination of critical pressure for the oblate ellipsoid of rotation. a and b - semi-axis of ellipsoid, $b<a$. In view of the fact that the Gaussian curvature of the oblate ellipsoid monotonically increases in proportion to approach/approximation to the equator together with radius of parallel ρ, minimum p is obtained with bulge along equator. At the equator

$$\rho=a,\quad R_1=a,\quad R_2=\frac{a}{b^2}.$$

Hence

$$p=\frac{2E\delta^2}{\sqrt{3(1-\nu^2)}}\,\frac{1}{\frac{a^2}{2}-b^2}.$$

In the case of strongly oblate ellipsoid (b<<a)

$$p \simeq \frac{4E\delta^2}{\sqrt{3}(1-\nu^2)a^2}.$$

It is substantial to note that the value of critical pressure with any of flattening does not descend below this value.

4. Loss of stability of convex hull of rotation under external pressure. In § 1 and p. 1 of present paragraph, was examined a question concerning the loss of stability of flat convex hull under external pressure. In this case, for value p of critical pressure, was found the formula

$$p = \frac{2E\delta^2}{\sqrt{3}(1-\nu^2)R_1R_2}.$$

Obtaining this result, we during determination by that produced by external pressure the work

$$A = p\,\Delta V$$

utilized an assumption about the flatness of shell. Specifically,, a change in the volume ΔV, limited by shell, we calculated according to the formula

$$\Delta V = \int\int \zeta\,dx\,dy,$$

~~gde~~ where ζ - component along the axis z of the bending field during the deformation of shell.

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This formula would be precise, if the bending field was perpendicular to plane xy. In actuality this can be considered carried out only near the center of bulge. In connection with this for unsloped shell, we can expect another result about the value of external critical pressure. Let us examine this question for the convex hulls of rotation.

Let us assume that the convex hull of rotation under the action of external pressure loses stability with the formation of the system of the dents, arrange/located along certain parallel (Fig. 27). Judging according to the result, obtained in p. 2, it is possible to think that this form of loss of stability can realize itself, if the dent greatly stretched along the parallel on which they are arrange/located. Just as in the case of internal pressure, a change (during deformation) of the volume, limited by shell, we will compose of two parts - ΔV_i and ΔV_e: ΔV_i - decrease of volume, connected directly with the education/formation of the regions of bulge, and ΔV_e is caused proximity of the planes of parallels, which limit the zone of the regions of bulge. We have

$$\Delta V_i = \frac{\pi\sigma}{\sqrt{ab}} (\lambda^4 + \mu^4 + 4\lambda^2\mu^2)\, n,$$

$$\Delta V_e = -\pi\rho^2 \frac{3\pi}{\Delta y} \lambda\mu (\lambda^2 + \mu^2)\, \sigma \cos\alpha.$$

Consequently, the conducted by the external pressure p work during the deformation of shell is equal to

$$A = \frac{\pi\sigma}{\sqrt{ab}} (\lambda^4 + \mu^4 + 4\lambda^2\mu^2)\, pn - \\ - \pi\rho^2 \frac{3\pi}{\Delta y} \sqrt{\frac{a}{b}} \lambda\mu (\lambda^2 + \mu^2)\, \sigma p \cos\alpha.$$

This formula differs only in terms of sign from the appropriate formula for the case of internal pressure.

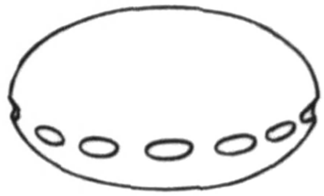

Fig. 27.

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As far as strain energy is concerned, for it remains the previous expression

$$U = \frac{2E\,\delta^2 2\pi \sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}\,(\lambda^4 + \mu^4 + 4\lambda^2\mu^2)\,\sigma n.$$

Just as in the case of internal pressure, from the condition of the equilibrium of the shell

$$\frac{d}{d\sigma}(U - A) = 0$$

we obtain the following relationship/ratio for the value of pressure p, received by shell with the bulge:

$$\frac{2E\,\delta^2 2\pi \sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}\,(\lambda^4 + \mu^4 + 4\lambda^2\mu^2)\,n - \\ - \frac{\pi}{\sqrt{ab}}\,(\lambda^4 + \mu^4 + 4\lambda^2\mu^2)\,np + \\ + \pi\rho^2\,\frac{3\pi}{\Lambda y}\,\sqrt{\frac{a}{b}}\,\lambda\mu\,(\lambda^2 + \mu^2)\,p\cos\alpha = 0.$$

This relationship/ratio by known method is simplified and, being

solved relative to p, it gives for it the following value:

$$p=\frac{4E\,\delta^2ab}{\sqrt{12}\,(1-\nu^2)}\,\frac{1}{-\frac{3a\rho\cos\alpha}{2}\,\vartheta^*+1},$$

where, as before,

$$\vartheta^*=\frac{\vartheta}{1+2\vartheta^2},$$

$$\vartheta=\frac{\lambda\mu}{\lambda^2+\mu^2}.$$

The small value p is obtained at the greatest in absolute value negative value ϑ^*, that is, with ϑ=-1/3.

If this value ϑ^* is substituted into formula for p, then we will obtain

$$p=\frac{4E\,\delta^2ab}{\sqrt{12}\,(1-\nu^2)}\,\frac{1}{\frac{a\rho}{2}\cos\alpha+1}.$$

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Application/use of this formula for slightly curved shells gives the value, which differs little from that obtained earlier

$$p=\frac{2E\,\delta^2ab}{\sqrt{3}\,(1-\nu^2)},$$

since for slightly curved shells $\alpha\approx\pi/2$ and, therefore, cos $\alpha\cong 0$.

Let us explain the form of the regions of bulge. So kA ϑ^*=-1/3,

then $\vartheta=-1/2$. Hence it follows that $\lambda=-\mu$. The region of bulge is assigned by the equation

$$x=\frac{\lambda+\mu}{\sqrt{a}}\cos\varphi, \quad y=\frac{\lambda-\mu}{\sqrt{b}}\sin\varphi.$$

With $\lambda=-\mu$ our ellipse degenerates in axis intercept y. Physically this means that the regions of bulge with the loss of stability of shell under external pressure are strongly elongated along parallel.

Just as in the case of internal pressure, formula for a critical load can be converted to the form

$$p=\frac{4E\delta^2}{\sqrt{12}(1-\nu^2)R_1R_2}\frac{1}{\frac{\rho^2}{2R_1R_2}+1},$$

or

$$p=\frac{2E\delta^2}{\sqrt{3}(1-\nu^2)}\frac{1}{\frac{\rho^2}{2}+R_1R_2}.$$

Let us recall that here R_1 and R_2 - main radii of curvature of shell along the parallel where occurs bulge, ρ - a radius of parallel.

As the application/appendix of the obtained result, let us examine the loss of stability of closed spherical shell of radius R. Here

$$R_1=R, \quad R_2=R.$$

Minimum p is obtained with ρ=R, that is, during education/formation indentation along equator. For the value of critical pressure, is obtained the formula

$$p = \frac{2E\delta^2}{\sqrt{3}\,(1-\nu^2)\,R^2} \cdot \frac{2}{3}.$$

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Thus, the obtained value composes 2/3 from the value, obtained for slightly curved shells.

Observation. According to data some experiments above spherical segments, described in A. S. Vol'mir's book [9], the bulge under external pressure begins at the edge of segment. It is possible to think that in this case occurs the loss of stability, examined in present point/item.

5. Loss of stability of covex shell of revolution during twisting. The shell of revolution, which is located under the action of the torsional moment, applied to the edge of shell, can lose stability with the education/formation of the regions of bulge, sloped toward meridian (Fig. 28). Let us examine a question concerning torque, calling the loss of stability of this form.

Approaching the deformed surface of shell by the isometric conversion of initial form, we will use the same considerations, as in p. 2 for the case of internal pressure. In view of the fact that

the regions of bulge are sloped toward meridian, occurs the twisting of shell to certain angle ε. Let us determine the value of this angle.

Let us take on surface the arbitrary point A out of the zone of the regions of bulge, but near this zone, and let us find the shift of this point along parallel during the deformation of shell in question. Let us conduct from point A perpendicular to parallel γ, where are arrange/located the centers of bulge, and let us designate through P the basis/base of this perpendicular. Just as in p. 2, we introduce Cartesian coordinates, after accepting point P in the origin of coordinates, tangential plane at this point for plane xy, and x axis it is directed along the meridian of surface.

Here $\tilde{\eta}$ - which interests us shift of point A. Then, assuming that value $\tilde{\eta}$ substantially affect only the region of the bulges, located near point P, we can write

$$\tilde{\eta}=\sum_{k}\eta(h,\ y_k).$$

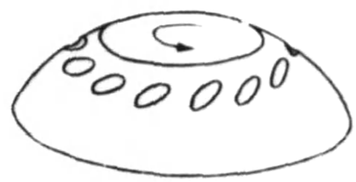

Fig. 28.

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Here $\eta(x, y)$ - are components along the axis y of the bending field, which corresponds to the region of bulge with center P, h - distance of point A from parallel γ, and y_k - coordinate of the centers of the regions of bulge, close to P.

let us examine the function $\eta(x, y)$. It is expressed as analytic function

$$\zeta=\frac{\alpha}{\omega^2}$$

on the formula

$$\eta=\alpha\sqrt{b}\,\mathrm{Re}\left\{-(u\sin\vartheta+v\cos\vartheta)-ie^{i\vartheta}\int\zeta\,dw\right\}.$$

Substituting in this formula $\zeta=\alpha/\omega^2$ and noting that

$$w=\lambda\omega+\frac{\mu}{\omega},$$

let us have

$$\eta = a\sqrt{b}\,\mathrm{Re}\left\{-\frac{(u\sin\vartheta + v\cos\vartheta)}{\omega^2} + \frac{i\lambda e^{i\vartheta}}{\omega} - \frac{i\mu e^{i\vartheta}}{\omega^3}\right\}.$$

For the low regions of bulge, the value $|\omega|$ is great. Therefore it is possible to count that

$$\eta = a\sqrt{b}\,\mathrm{Re}\left\{-\frac{(u\sin\vartheta + v\cos\vartheta)}{\omega^2} + \frac{i\lambda e^{i\vartheta}}{\omega}\right\}.$$

On the same reason

$$w \simeq \lambda\omega.$$

Thus,

$$\eta = a\lambda^2\sqrt{b}\,\mathrm{Re}\left\{-\frac{(u\sin\vartheta + v\cos\vartheta)}{w^2} + \frac{ie^{i\vartheta}}{w}\right\}.$$

Let us pass from the variables u, v to x, y. We have

$$x = \frac{1}{\sqrt{a}}(u\cos\vartheta - v\sin\vartheta),$$
$$y = \frac{1}{\sqrt{b}}(u\sin\vartheta + v\cos\vartheta),$$
$$x\sqrt{a} + iy\sqrt{b} = (u + iv)e^{i\vartheta} = we^{i\vartheta},$$
$$ax^2 + by^2 = u^2 + v^2 = |w|^2.$$

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With the aid of these relationship/ratios the expression for η is converted to the form

$$\eta = 2\lambda^2\alpha\sqrt{b}\,\frac{2y^3b\sqrt{b}\cos 2\theta - (3xy^2b\sqrt{a} + x^3a\sqrt{a})\sin 2\theta}{(ax^2+by^2)^2}.$$

Substituting this value in expression $\tilde{\eta}$, let us have

$$\tilde{\eta} = 2\lambda^2\alpha\sqrt{b}\sum_k \frac{2y_k^3b\sqrt{b}\cos 2\theta - \left(3hy_k^2b\sqrt{a} + h^3a\sqrt{a}\right)\sin 2\theta}{\left(ah^2+by_k^2\right)^2}.$$

In view of the fact that the regions of bulge are arrange/located symmetrically relative to point A, summation over first term can be drop/omitted, after accepting

$$\tilde{\eta} = -2\lambda^2\alpha\sqrt{b}\sin 2\theta\sum_k \frac{3hy_k^2b\sqrt{a} + h^3a\sqrt{a}}{\left(ah^2+by_k^2\right)^2}.$$

It is further, as in p. 2, in right side of addition we pass to integral. Specifically,, set/assuming

$$\Delta y = y_k - y_{k-1},$$

let us have

$$\tilde{\eta} = -\frac{2\lambda^2\alpha\sqrt{b}\sin 2\theta}{\Delta y}\int_{-\infty}^{\infty}\frac{3hy^2b\sqrt{a} + h^3a\sqrt{a}}{(ah^2+by^2)^2}\,dy.$$

Introducing new the variable t

$$y = h\sqrt{\frac{a}{b}}\,t,$$

we will obtain

$$\tilde{\eta} = -\frac{2\lambda^2\alpha\sin 2\theta}{\Delta y}\int_{-\infty}^{\infty}\frac{3t^2+1}{(1+t^2)^2}\,dt.$$

We have

$$\int_{-\infty}^{\infty} \frac{3t^2+1}{(1+t^2)^2}\,dt = 2\pi.$$

Therefore

$$\tilde{\eta} = -\frac{4\pi\lambda^2 a \sin 2\vartheta}{\Delta y}.$$

Introducing here

$$a = -\frac{3\mu}{\lambda}(\lambda^2+\mu^2)\,\sigma,$$

we will obtain final formula for $\tilde{\eta}$

$$\tilde{\eta} = \frac{12\pi\lambda\mu\,(\lambda^2+\mu^2)\,\sigma \sin 2\vartheta}{\Delta y}.$$

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As one would expect, the shift $\tilde{\eta}$ of point A does not depend on distance of h.

Now it is possible to determine angle of twist ε. If we designate through ρ a radius of the parallel γ, along which are arrange/located the regions of bulge, then

$$\varepsilon = \frac{2\tilde{\eta}}{\rho} = \frac{24\pi\lambda\mu\,(\lambda^2+\mu^2)\,\sigma \sin 2\vartheta}{\rho\,\Delta y}.$$

Produced by torque/moment M work with bulge is equal to

$$A = M\varepsilon = \frac{24\pi\lambda\mu(\lambda^2+\mu^2)\sigma M \sin 2\vartheta}{\rho\,\Delta y}.$$

Let us find now strain energy U. In § 1 for strain energy $\bar{U}$ per the unit of the length of the boundary of bulge is obtained the formula (page 109)

$$\bar{U} = \frac{2E\delta^2\alpha^2 hk}{\sqrt{12}(1-\nu^2)}.$$

Here α - angle between the osculating plane curve γ, that limits the region of bulge, and by the tangential planes of surface, k - curvature curved γ, h - sagging/deflection in the region of bulge.

In the coordinate system which is introduced above, our surface near point P is assigned by the equation

$$z = \frac{1}{2}(ax^2 + by^2).$$

Curve γ is assigned on it by the equations

$$u = (\lambda+\mu)\cos\varphi, \quad v = (\lambda-\mu)\sin\varphi.$$

The variables u, v are connected with by x, y formulas

$$x = \frac{1}{\sqrt{a}}(u\cos\vartheta - v\sin\vartheta),$$

$$y = \frac{1}{\sqrt{b}}(u\sin\vartheta + v\cos\vartheta).$$

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For the low region of bulge, it is possible to count

$$\alpha = \frac{k_n}{k},$$

where k - curvature curved γ, k_n - normal curvature of initial surface in direction γ. Common/general/total expression of the normal curvature

$$k_n = \frac{a\,dx^2 + b\,dy^2}{dx^2 + dy^2}.$$

Substituting here the values

$$x = \frac{1}{\sqrt{a}}\{(\lambda+\mu)\cos\varphi\cos\vartheta - (\lambda-\mu)\sin\varphi\sin\vartheta\},$$
$$y = \frac{1}{\sqrt{b}}\{(\lambda+\mu)\cos\varphi\sin\vartheta + (\lambda-\mu)\sin\varphi\cos\vartheta\},$$

we will obtain normal surface curvature in direction γ:

$$k_n = \frac{ax'^2 + by'^2}{x'^2 + y'^2},$$

where the differentiation is fulfilled according to variable ϑ.

For the low region of bulge the curvature γ can be determined according to its projection on plane xy. We will obtain

$$k = \frac{|x''y' - y''x'|}{(x'^2 + y'^2)^{3/2}}.$$

element of arc to curve γ is equal to

$$ds = (x'^2 + y'^2)^{1/2}\, d\varphi.$$

Let us calculate now the integral

$$\int_\gamma \alpha^2 k\, ds = \int_0^{2\pi} \frac{(ax'^2 + by'^2)^2}{|x''y' - y''x'|}\, d\varphi.$$

We have

$$|x''y' - y''x'| = \frac{1}{\sqrt{ab}}(\lambda^2 - \mu^2),$$

$$ax'^2 + by'^2 = (\lambda + \mu)^2 \sin^2\varphi + (\lambda - \mu)^2 \cos^2\varphi.$$

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Substituting these values in integrand, we find

$$\int_\gamma \alpha^2 k ds = \frac{2\pi\sqrt{ab}}{\lambda^2 - \mu^2}(\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

Total energy of deformation for one region of bulge is equal to

$$U_1 = \int_\gamma \bar{U}\, ds_\gamma = \frac{2E\delta^2 h}{\sqrt{12}(1-\nu^2)} \frac{2\pi\sqrt{ab}}{\lambda^2 - \mu^2}(\lambda^4 + \mu^4 + 4\lambda^2\mu^2).$$

Substituting here

$$h = (\lambda^2 - \mu^2)\sigma,$$

we will obtain

$$U_1 = \frac{2E\delta^2 2\pi\sqrt{ab}}{\sqrt{12}(1-\nu^2)}(\lambda^4 + \mu^4 + 4\lambda^2\mu^2)\sigma.$$

Strain energy on all n to the regions of bulge is equal to

$$U=\frac{2E\delta^2 2\pi\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}\,(\lambda^4+\mu^4+4\lambda^2\mu^2)\,\sigma n.$$

Now from the condition of the equilibrium of the shell

$$\frac{d}{d\sigma}(U-A)=0$$

we obtain relationship/ratio for torque/moment M, calling the loss of stability of the shell:

$$\frac{2E\delta^2 2\pi\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}\,(\lambda^4+\mu^4+4\lambda^2\mu^2)\,n-$$
$$-\frac{24\pi\lambda\mu\,(\lambda^2-\mu^2)\,M\cdot\sin 2\vartheta}{\rho\,\Delta y}=0.$$

Hence, noting that

$$n\,\Delta y=2\pi\rho,$$

we will obtain

$$\frac{\pi\rho^2 E\delta^2\sqrt{ab}}{\sqrt{12}\,(1-\nu^2)}\,(1+2\varepsilon^2)-3\varepsilon M\sin 2\vartheta=0,$$

where

$$\varepsilon=\frac{\lambda\mu}{\lambda^2+\mu^2}.$$

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The small value M is obtained at $\varepsilon=1/2$ and ϑ=45°. This value is

determined from the formula

$$M = \frac{\pi\rho^2 E\delta^2 \sqrt{ab}}{\sqrt{12(1-\nu^2)}}$$

or

$$M = \frac{\pi\rho^2 E\delta^2}{\sqrt{12(1-\nu^2)}\sqrt{R_1 R_2}}.$$

Here R_1 and R_2 - main radii of curvature along the parallel where occurs bulge, ρ - a radius of this parallel.

In conclusion let us note that the loss of stability under the action of the torsional moment M is accompanied by the education/formation of strongly elongated dents $(\varepsilon = 1/2)$, sloped toward meridian at angle of θ=45°.

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Chapter Three.

CYLINDRICAL SHELLS DURING SUPERCRITICAL DEFORMATIONS.

Among the shells, which are encountered in real construction/designs, most widely used are the shells, which have the form of developable surfaces. To a considerable degree this is explained by simplicity of their obtaining from sheet material. Among the developable shells most widely used, perhaps, are cylindrical. In connection with this the study of the strength properties of these shells is of great practical interest.

Cylindrical shell as the element of construction/design can work under varied conditions of loading. Among them basic are: axial compression, external pressure and twisting. Usually the decomposition of shell occurs as a result of the loss of stability. The supercritical deformations which in this case appear, lead to stress concentration in the specific zones, that also leads finally to decomposition. In connection with this appears the natural task of

the determination of the elastic states of shell during supercritical deformations.

In present chapter we investigate the supercritical elastic states of cylindrical shells in basic load cases: axial compression (§1), external pressure (§2) and twisting (§3). Just as in the case of strictly convex hulls (Chapter 1), this investigation will be based on principle A. The result of investigation will be the complete description of supercritical elastic state, in particular, the determination of lower critical loads.

The methods, developed in this chapter, can be used also to the case of conical shells. In regard to this see works [10], [11] V. I. Babenko in [12] V. V. Mixailova.

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§1. Supercritical deformations of cylindrical shells during axial compression.

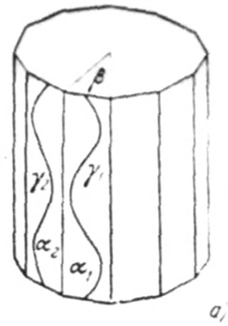
As noted above, proposed study of the supercritical elastic states of cylindrical shell will be based on the use of principle A. The application/use of this principle assumes the investigation of the possible isometric transformations of the initial surface of

shell. Therefore we will begin our study with the study of these transformations.

1. Special isometric transformation of cylindrical surface. Experiment shows that the supercritical deformation of the geometrically modern cylindrical shell during axial compression possesses the specific correctness of structure. Specifically,, is observed the distinct periodicity of the form of the deformed surface on circumference and height/altitude of shell. In connection with this we narrow our task and are limited to the examination of the isometric transformations, which possess the correctness of structure indicated.

Let us take correct prism with even number of sides (2n) and will conduct on one of its side faces α_1 the arbitrary smooth curve γ_1, which is unambiguously design/projected for the axle/axis of prism (Fig. 29a). It is reflected curve γ_1 mirror in plane β, passing through the lateral edge face α_1 and the axle/axis of prism. In this case, to obtain the curve γ_2, which lies at side face α_2, adjacent α_1. Then we analogously plot a curve γ_3 in face α_3, adjacent α_2, and so forth. So in each face α_i let us construct curve γ_i.

Let us conduct now through curves γ_1 and γ_2 the cylindrical surface Z_{12} with generatrices, perpendicular to plane β. Let us analogously construct the cylindrical surfaces Z_{23}, Z_{34} and so forth.



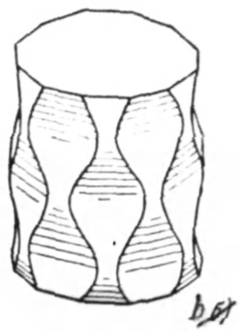


Fig. 29.

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Surfaces Z_{12}, Z_{23}, ... compose the tubular surface of Z, everywhere smooth, besides fin/edges γ_1, γ_2, ... (Fig. 29b). It is confirmed that constructed thus surface Z is isometric to cylinder.

In order this to demonstrate, let us show first that surface Z is locally isometric plane, i.e., that each point of this surface has vicinity, isometric to the piece of plane. This is obvious for the points, which do not lie on fin/edges γ_i of surface. Let us examine

the points on one of curves γ_1, γ_2.

It is reflected surface Z_{23} mirror in the plane of face α_2. Obtained in this case surface Z^*_{23} is the continuation of surface Z_{12} beyond edge γ_2 in the form of cylindrical surface. The surface, comprised of Z_{12} and Z^*_{23} as is cylindrical, by lengthwise curve γ_2 it is locally isometric plane. Hence it follows that along this curved the surface, comprised of Z_{12} and Z_{23} and, consequently, also Z, is also locally isometric plane.

The given construction/design allows to easily conclude also that the closed broken line $\bar{\gamma}$, comprised of linear generator surfaces Z_{12}, Z_{23}, ..., i.e., the intersection of surface of Z with plane, perpendicular to axle/axis of prism, is closed geodetic. The two considerations presented (concerning the local isometry of plane and the closed geodetic) it is sufficient in order to conclude about the isometry of surface Z to circular cylinder.

Let us determine a radius of the cylinder, isometric of surface Z. For this, let us examine the intersection of surface of Z with the plane, perpendicular to the axle/axis of prism. As it was shown, obtained in this case closed broken line $\bar{\gamma}$ is geodetic, and therefore its length L is connected with radius R of cylinder by the relationship/ratio

$$L = 2\pi R.$$

Broken line $\bar{\gamma}$ is entered in the correct 2n- angle plate $\bar{\gamma}_0$, on which the plane at which lie/rests $\bar{\gamma}$, intersects the lateral surface of prism (Fig. 30). Since both sides $\bar{\gamma}$ with the sides of polygon $\bar{\gamma}_0$ form equal angles $(\pi/2n)$, then independent of form broken line $\bar{\gamma}$ has always some and the same perimeter L, equal to the perimeter of correct 2n- angle plate with apex/vertexes in the middles of the sides of polygon $\bar{\gamma}_0$.

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Now does not compose the work to find the perimeter of broken line $\bar{\gamma}$ and, consequently, also radius R of the cylinder, isometric of surface Z.

The given consideration permits to draw an important conclusion. Specifically,, a radius of cylinder, isometric Z, does not depend on which was undertaken curve γ on the face of prism α_1 during the construction of surface of Z.

Let us conduct through the axle/axis of prism and one of its lateral edges plane. It will cross surface of Z according to certain curve $\tilde{\gamma}$. On the surface of the circular cylinder to which is isometric surface Z, curved $\tilde{\gamma}$ on isometry corresponds linear generator. Consequently, length curved $\tilde{\gamma}$ is equal to the

height/altitude of cylinder, isometric Z, and it does not depend on that, through which of the lateral edges is carried out the secant plane, which is determining curve $\tilde{\gamma}$.

Let us assume now that curved γ_1 on the side face α_1, with the aid of which construction/design described above is obtained surface Z, by arbitrary form is deformed, but so that the length curve $\tilde{\gamma}$ is retained. In this case, surface Z is also deformed. And since a radius and a height/altitude of cylinder, isometric Z, does not change, then this deformation is geometric bending. With the aid of this bending we will approach elastic deformation of cylindrical shell in supercritical stage.

In connection with the application/use of principle A to the investigation of the supercritical elastic states of cylindrical shells during axial compression us must examine the functional

$$W = U - A$$

on many all isometric transformations of cylindrical surface, which possess the periodicity of structure.

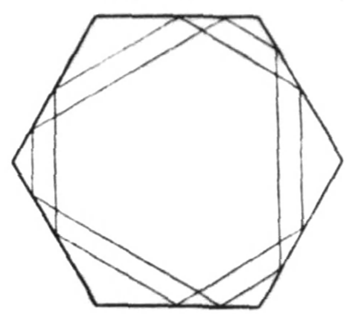

Fig. 30.

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If as curve γ, with the aid of which is constructed in a manner described above surface Z, to take periodic curve, then surface Z will possess this periodicity. On arises the question, any whether isometric to cylinder the surface, which possesses the periodicity of structure, can be constructed thus. Let us show that this is really/actual thus.

Let certain surface Z possess the periodicity of structure on height/altitude and in circular direction. It is required to show that it is obtained construction/design described above. Retaining the succession of designations, let us call/name β one of the radial planes of the symmetry of surface Z (Fig. 31). This plane intersects surface according to certain curve $\tilde{\gamma}$. P - arbitrary point in this curve. Since surface Z developing, then through each of its points,

in particular through point P, passes linear generator $g(P)$. If we assume for simplicity that the surface Z does not contain flat/plane pieces, then linear generator $g(P)$, passing through point P, will be only.

In view of the symmetry of surface Z relative to plane β, linear generator $g(P)$, being only, lie/rests either at plane β or it is perpendicular to this plane. The first possibility is eliminated, since otherwise the transition of cylindrical surface into surface of Z is not accompanied by axial compression. Thus, linear generator surfaces at the points lines $\tilde{\gamma}$ must be perpendicular to plane β, and therefore parallel to one another. But this that means that surface Z near line $\tilde{\gamma}$ must be cylindrical surface, with generatrices, perpendicular to the plane of symmetry β. The same structure has surface of Z from opposite side near line $\tilde{\gamma}_1$.

Since surface Z has n of planes of symmetry, then it must consist of 2n cylindrical surfaces, with generatrices, perpendicular to these planes.

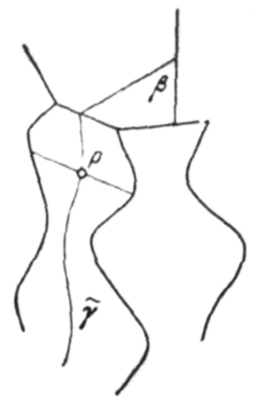

Fig. 31.

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Being so it is arranged, surface Z must have special lines - fin/edge. γ - one of such fin/edges, and Z_1, Z_2 - cylindrical surfaces which intersect on this fin/edge (Fig. 32). Since surface Z is locally isometric plane, then surfaces Z_1 and Z_2 cannot be completely arbitrary. Let us establish communication/connection between them.

Let $\tilde{\gamma}$ - intersection of surface Z_1 with the plane of symmetry Z, perpendicular to its generatrices. Curve $\tilde{\gamma}$ is geodetic line. Let us characterize the position of the arbitrary point P of curve γ distance u(s) of this point from $\tilde{\gamma}$ on generatrix (s - arc lengthwise $\tilde{\gamma}$, Fig. 31). Since surface Z is locally isometric plane, then the sum

of the geodetic curvatures of line γ on surfaces Z_1 and Z_2 must be equal to zero. In the assigned/prescribed surface Z_1 and the direction of generatrices surfaces Z_2, this condition gives certain differential second order equation for function u(s):

$$u'' = \varphi(u', s). \qquad (*)$$

From the uniqueness of the solution of this equation, it follows that curve γ is determined unambiguously (on surface Z_1), if is assign/prescribed its any point and direction in it.

In view of the periodicity of the structure of surface of Z on height/altitude, on line γ will be located such point P_0, at which the tangent to it is parallel to the axle/axis of the surface (axle/axis we we call straight line along which intersects the planes of symmetry). Let us conduct through point P_0 the plane α, parallel to the axle/axis of surface, so that generatrices of surfaces Z_1 and Z_2, that proceed from point P_0, would compose equal angles with plane α and would be arrange/located along its one side. Let γ' - it is curved on which the plane α intersects surface Z_1 and its continuation for curve γ. Curve γ' - satisfies equation (*). The corresponding surface Z'_2 is constructed by mirror reflection in the plane α of that part of surface Z_1 and of its continuation, that is located beyond curve γ'.

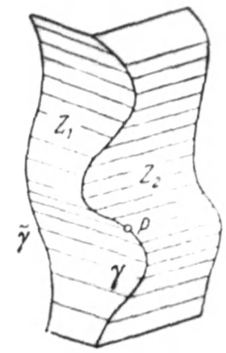

Fig. 32.

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Since curves γ and γ' have common point (P_0) and direction in it, then they coincide. Hence it follows that surface Z is obtained construction/design described above. Plane α is one of the faces of prism.

In connection with the computation of the functional

$$W = U - A$$

during the isometric transformations of the initial surface of shell, let us determine some values for that constructed, isometric to cylinder, surface Z.

Let us designate α one of the faces of prism, into which is entered surface Z. The fin/edge of this surface, which lies at face

α, let us designate γ. Let us conduct through the axle/axis of prism and the lateral edge of face α plane β, and its intersection with surface of Z let us designate $\tilde{\gamma}$. Curve $\tilde{\gamma}$ is normal section of surface of Z, perpendicular to generatrices.

Let us introduce in the plane of face α the rectangular Cartesian system of coordinates x, y, after accepting for x axis straight line, parallel to the lateral edges of face and which passes in the middle between the mini ones, but for y axis - the direct/straight, perpendicular x axis. Let in these coordinates the fin/edge γ of surface Z be assigned by the equation

$$y = y(x).$$

In plane β, let us also introduce the rectangular Cartesian coordinate system, after accepting for x and y axes of the projection of the coordinate axes, introduced in plane α. In these coordinates normal section $\tilde{\gamma}$ of surface of Z is assigned by the equation

$$y = \tilde{y}(x) = \sin\frac{\pi}{2n}\, y(x).$$

We will assume n sufficient to large ones, and therefore it is possible to count that

$$\tilde{y}(x) \simeq \frac{\pi}{2n}\, y(x).$$

On known formula the curvature of the fin/edge γ of surface Z is equal to

$$k = \frac{|y''|}{(1+y'^2)^{3/2}}.$$

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Normal surface curvature Z in the section/cut, perpendicular to generatrices, will be

$$\tilde{k} = \frac{\tilde{y}''}{\left(1 + \tilde{y}'^2\right)^{3/2}}.$$

Assuming n large, we will drop/omit term $\tilde{y}'^2$ in the denominator of this formula. Then we obtain

$$\tilde{k} = \tilde{y}''.$$

Or, introducing instead of $\tilde{y}$ function y,

$$\tilde{k} = \frac{\pi}{2n} y''.$$

Let us determine the angle ϑ, which forms the plane of the fin/edge γ of surface Z with tangential planes. For this purpose, the system of coordinates xy in plane β let us supplement to the three-dimensional system of coordinates xyz. In this coordinate system, the angular coefficients of the plane of fin/edge γ, i.e., plane α, will be 0, 1,$-\pi/2n$, the angular coefficients of the tangential plane of surface will be $\pi y'/2n$, 1, 0. The angle between planes is equal to the angle between vectors (0, 1,$-\pi/2n$), ($\pi y'/2n$, 1, 0). Hence for an angle ϑ with large n, is obtained the following value:

$$\vartheta = \frac{\pi}{2n} \sqrt{1 + y'^2}.$$

Let us find axial compression during the isometric transformation of circular cylinder into surface of Z. Let us call on cylinder region G, situated between two section/cuts, perpendicular

to axle/axis and distant from each other up to distance of b. Isometric by it region Z_G on surface of Z is limited by two planes, perpendicular to the axle/axis of prism and distant from each other at certain distance b'. The axial compression, under discussion, to eat difference b-b'=Δb.

Height/altitude b of belt/zone G on cylinder, as shown above, is equal to length curved $\tilde{\gamma}$ on surface Z_G.

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Consequently,

$$b=\int \sqrt{1+\tilde{y}'^2}\,dx,$$

where the integration is fulfilled on the height/altitude of belt/zone Z_G of surface Z, and

$$\int dx = b'.$$

Assuming, as before $\tilde{y}'^2$ small, we can write

$$b \simeq \int \left(1+\frac{\tilde{y}'^2}{2}\right) dx = b' + \frac{\pi^2}{8n^2}\int y'^2\,dx.$$

Hence

$$\Delta b \simeq \frac{\pi^2}{8n^2}\int y'^2\,dx.$$

Let us assume now that function y(x), that assigns curve γ, is periodic and even. m - number of complete waves curved γ. Let us conduct through the apexes of curve γ plane, perpendicular to the axle/axis of prism, and will call/name their horizontal section/cuts.

Furthermore, let us conduct the half-planes through the axle/axis of prism and its side edge - radial sections. These planes divide/mark off surface of Z on $4mn$ the equal regions Q, each of which is isometric to rectangle (Fig. 33). Let us determine the sides of this rectangle depending on the parameters of cylinder, isometric Z, i.e., a radius of basis/base, height/altitude, numbers m and n.

If the height/altitude of cylinder L, then, obviously, the height/altitude of rectangle Q is equal to

$$b = \frac{L}{2m}.$$

If a radius of cylinder R, then the width of rectangle Q is equal to

$$a = \frac{\pi R}{n}.$$

On isometry to the separation of surface of Z on region Q corresponds the separation of cylinder, isometric Z, on $4mn$ rectangular regions by the planes, perpendicular to axle/axis, distant from each other at a distance of b, and by the radial half-planes, which separate complete angle with the axle/axis of cylinder on 2n equal parts.

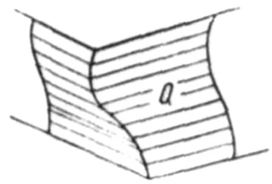

Fig. 33.

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2. Strain energy of shell. Investigation of supercritical deformations during axial compression. According to principle A, the determination of the supercritical elastic states of shell is reduced to the examination of variational problem for functional W=U-A which is determined during the isometric transformations of the initial form of shell. As shown in p. 1, the class of the isometric transformations during which one should examine functional W in the case of the axial compression of cylindrical shells, it becomes narrow to surfaces of the type Z. In connection with this we will determine strain energy U on surfaces of this type.

In p. 1, we broke surface of Z on $4mn$ the congruent regions Q, correctly located in 2m belt/zones on 2n regions in each belt/zone. Regions Q are isometric to rectangle with basis/base $a=\pi R/n$, height/altitude $b=L/2m$ and are comprised of two cylindrical

surfaces, which adjoin on fin/edge γ. In order to find the strain energy of an entire shell, it suffices to find it in one region and result to multiply on $4mn$.

Strain energy of region Q consists of two parts: U_Q - strain energy over basic surface and U_γ - strain energy along fin/edge. Energy U_Q is determined by the curvature of initial cylindrical form into form of Z everywhere, besides fin/edge γ. Per the unit surface area, it is determined from the formula

$$\overline{U}_Q = \frac{D}{2}(\Delta k_1^2 + \Delta k_2^2 + 2\nu\Delta k_1\Delta k_2).$$

Here Δk_1 and Δk_2 - main changes in the normal curvatures with deformation of surface indicated, but D - the flexural rigidity of shell, i.e.,

$$D = \frac{E\delta^3}{12(1-\nu^2)},$$

where δ - thickness of shell, E - modulus of elasticity, and ν - Poisson ratio.

For the normal curvatures k_1 and k_2 of initial cylindrical surface, we have

$$k_1 = 0, \quad k_2 = \frac{1}{R}.$$

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In the appropriate directions normal surface curvatures Z are equal

to

$$k_1 = \pm \frac{\pi}{2n} y'', \quad k_2 = 0.$$

(Normal curvatures in adjacent radial normal sections of surface, which form angle π/n, are equal in magnitude, but they are opposite on sign. By this is explained ambiguity in expression k_1). Thus, during the deformation of surface in question it will be

$$\Delta k_1 = \pm \frac{\pi}{2n} y'', \quad \Delta k_2 = -\frac{1}{R}.$$

Hence

$$U_Q = \frac{D}{2} \iint_{(Q)} \left\{ \left(\frac{\pi y''}{2n} \right)^2 + \frac{1}{R^2} \pm 2\nu \left(\frac{\pi y''}{2n} \right) \frac{1}{R} \right\} dx\, dy.$$

Fulfilling integration for y, we will obtain

$$U_Q = \frac{Da}{2} \int_{(b)} \left(\frac{\pi y''}{2n} \right)^2 dx + \nu D \int_{(b)} \frac{\pi}{2nR} y'' (a' - a'')\, dx + \text{const},$$

where a' and a'' - length of cuts which form surfaces of Z into regions Q, divided by fin/edge γ. With large n

$$a' \simeq \frac{a}{2} - y, \quad a'' \simeq \frac{a}{2} + y.$$

Therefore

$$\int_{(b)} y'' (a' - a'')\, dx = -2 \int_{(b)} y y''\, dx.$$

Fulfilling integration in parts and noting that y' at the ends of the interval of integration (b) is equal to zero, we will obtain

$$\int_{(b)} y'' (a' - a'')\, dx = 2 \int_{(b)} y'^2\, dx.$$

Consequently,

$$U_Q = \frac{D\pi^2 a}{8n^2} \int_{(b)} y''^2\, dx + \frac{\nu D\pi}{nR} \int_{(b)} y'^2\, dx + \text{const}.$$

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Strain energy U_γ along fin/edge γ is determined from the formula

$$U_\gamma = U^0_\gamma + \Delta U_\gamma,$$

where

$$U^0_\gamma = \int\limits_\gamma cE\delta^{5/2}\alpha^{5/2}k^{1/2}\,ds,$$

$$\Delta U_\gamma = \frac{E\delta^3}{6(1-\nu^2)}\int\limits_\gamma \alpha\left(-k_n + \frac{k_i + k_e}{2}\right)ds.$$

Here k - curvature curved γ, α - angle between the plane of fin/edge γ and the tangential planes of surface Z along fin/edge; k_i and k_e - normal surface curvatures Z in the direction, perpendicular to fin/edge, k_n - the normal curvature of initial surface in the appropriate direction. Integration is fulfilled according to arc s by curve γ.

Let us find value ΔU_γ. Normal curvatures k_i and k_e are equal in magnitude and are opposite on sign. Therefore

$$k_i + k_e = 0$$

and, therefore,

$$\Delta U_\gamma = -\frac{E\delta^3}{6(1-\nu^2)}\int\limits_\gamma \alpha k_n\,ds.$$

Angle

$$\alpha = \frac{\pi}{2n}\left(1 + y'^2\right)^{1/2}.$$

On the Euler formula the normal curvature

$$k_n = \frac{1}{R}\sin^2\vartheta,$$

where ϑ - angle which composes linear generator of initial cylindrical surface with direction by curve, appropriate on isometry

γ, i.e.,

$$\cos\vartheta = \frac{1}{(1+y'^2)^{1/2}}.$$

Finally, the cell/element of arc to curve γ is equal to

$$ds = (1+y'^2)^{1/2}\,dx.$$

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Substituting the value α, k_n and ds into formula for ΔU_γ, we will obtain

$$\Delta U_\gamma = -\frac{E\delta^3}{12(1-\nu^2)}\frac{\pi}{nR}b.$$

Let us calculate U_γ^0. Taking into account of expression for α, ds and

$$k = \frac{|y''|}{(1+y'^2)^{3/2}},$$

we will obtain

$$U_\gamma^0 = cE\delta^{5/2}\left(\frac{\pi}{2n}\right)^{5/2}\int_\gamma |y''|^{1/2}(1+y'^2)\,dx.$$

Total energy of deformation U_1 on region Q is obtained by the addition of values U_Q, U_γ^0 and ΔU_γ:

$$U_1 = \frac{D\pi^2 a}{8n^2}\int_{(b)} y''^2\,dx + \frac{\nu D\pi}{nR}\int_{(b)} y'^2\,dx + $$
$$+ cE\delta^{5/2}\left(\frac{\pi}{2n}\right)^{5/2}\int_{(b)} |y''|^{1/2}(1+y'^2)\,dx + \text{const.}$$

Of regions Q, situated between two planes, passing through the adjacent apex/vertexes of fin/edge γ it is perpendicular to the axle/axis of prism, is formed circular belt/zone with a height/altitude of b. Subsequently to us it will conveniently examine

not entire shell, but this belt/zone. The strain energy U of shell within the belt/zone indicated is obtained by the multiplication of strain energy in region Q on 2n - number of regions within belt/zone, i.e.,

$$U=\frac{D\pi^2a}{4n}\int\limits_{(b)} y''^2\,dx+\frac{2\nu D\pi}{R}\int\limits_{(b)} y'^2\,dx+ + 2cnE\delta^{5/2}\left(\frac{\pi}{2n}\right)^{5/2}\int\limits_{(b)} |y''|^{1/2}(1+y'^2)\,dx+\mathrm{const}.$$

Let us find the produced by external load work during the axial compression of shell under load p. The axial compression of the belt/zone in question is equal

$$\Delta b=\frac{\pi^2}{8n^2}\int\limits_{(b)} y'^2\,dx.$$

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Hence the work

$$A=2\pi R\,\delta p\,\frac{\pi^2}{8n^2}\int\limits_{(b)} y'^2\,dx.$$

Into the expression of the strain energy of shell U and of work A, produced by axial compression, enter the integral parameters m and n, the which characterize periodicity structures of surface of Z. In order to define these parameters, we let us assume that the character of the periodicity of the sagging/deflections of shell is retained during entire time of supercritical deformation and, therefore, it remains the same as at the moment of loss of stability.

In the linear theory of shells, it is proven, that the normal sagging/deflection w of the cylindrical shell of radius R at the moment of loss of stability satisfies the differential equation

$$\frac{D}{\delta}\Delta\Delta\Delta\Delta w+\frac{E}{R^2}\frac{\partial^4 w}{\partial x^4}+p\Delta\Delta\left(\frac{\partial^2 w}{\partial x^2}\right)=0. \qquad (*)$$

Here x and y - curvilinear coordinates: x - on generatrix, y - according to the circular section/cut, perpendicular to axle/axis; D - rigidity of shell to curvature, p - critical load, and

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

- operator of Laplace. Analyzing this equation, they come to the conclusion about the fact that the sagging/deflection of shell at the moment of loss of stability under the condition for hinged support for edges takes the form

$$w=c\sin\frac{2\pi mx}{L}\sin\frac{ny}{R}.$$

Substituting this expression in equation (*), we will obtain the relation between the parameters of wave formation m, n and critical load p:

$$\frac{D}{\delta}\left(\frac{4\pi^2m^2}{L^2}+\frac{n^2}{R^2}\right)^4+\frac{E}{R^2}\left(\frac{2\pi m}{L}\right)^4-p\left(\frac{4\pi^2m^2}{L^2}+\frac{n^2}{R^2}\right)^2\left(\frac{2\pi m}{L}\right)^2=0.$$

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Let us introduce instead of m and n the parameters ξ and η

$$\xi=\frac{Ln}{2\pi Rm},\quad \eta=\frac{n^2\delta}{R}.$$

Then, set/assuming

$$p = \bar{p} E \frac{\delta}{R},$$

let us have

$$\bar{p} = \frac{1}{12(1-\nu^2)} \frac{(1+\xi^2)^2}{\xi^2} \eta + \frac{\xi^2}{(1+\xi^2)^2 \eta}.$$

The small value $\bar{p}$ answers upper critical load. It is obtained, if the parameters ξ and η are connected by the relationship/ratio

$$\frac{\xi^2}{(1+\xi^2)^2 \eta} = \frac{1}{\sqrt{12(1-\nu^2)}}. \qquad (**)$$

Then

$$\bar{p}_e = \min \bar{p} = \frac{1}{\sqrt{3(1-\nu^2)}} \simeq 0,6.$$

Relationship/ratio (**) is not determined ξ and η, and consequently, m and n are unambiguous. However, as is shown experiment, shell loses stability in such a way that value $\xi \simeq 1$. If we for ξ accept this value, then relationship/ratio (**) determines η, and that means m and n. In particular, for n is obtained the following formula:

$$n = \frac{1}{2} \sqrt[4]{12(1-\nu^2)} \sqrt{\frac{R}{\delta}}.$$

When $\nu = 0,3$

$$n \simeq 0,91 \sqrt{\frac{R}{\delta}}.$$

After assuming that the periodicity of the sagging/deflections of shell is retained during supercritical deformation, and therefore it the same, as at the moment of loss of stability, we must consider that size/dimensions a, b of region Q of shell are identical, since

$$\frac{a}{b} = \xi = 1,$$

while parameter n is determined from formula indicated above.

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The true form of shell during the supercritical deformation, which is accompanied by axial compression Δb, is determined from the condition of the minimum of functional U(y) in the class of functions y(x), that satisfy the condition

$$\Delta b = \frac{\pi^2}{8n^2} \int_{(b)} y'^2\,dx.$$

For solving this variational problem, it is advisable to pass to dimensionless variable $\bar{x}$ and $\bar{y}$, set/assuming

$$x = \frac{b}{2}\,\bar{x}, \quad y = \frac{2n}{\pi}\sqrt{bh}\,\bar{y},$$

where h it designates the axial compression of shell (i.e. h=Δb). In new the variables $\bar{x}$, $\bar{y}$ the feature above which we for simplicity of recording lower, it will be

$$U = \frac{2E\delta^3 anh}{3(1-\nu^2)b^2} \int_{-1}^{1} y''^2\,dx + \frac{4\nu E\delta^3 hn^2}{3\pi(1-\nu^2)R} +$$
$$+ 2ncE\delta^{5/2}\left(\frac{\pi}{2n}\right)^2 (bh)^{1/4} \int_{-1}^{1} |y''|^{1/2}\left(1 + \frac{16n^2 h}{\pi^2 b}\,y'^2\right) dx + \text{const}.$$

Then the determination of the form of shell during supercritical deformation is reduced to the determination of function y(x), that realizes the minimum of functional J(y) under the condition

$$\int_{-1}^{1} y'^2\,dx = 1.$$

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Relative to function $y(x)$, for which functional J reaches the minimum, it is substantial to note that the graph of this function has point of inflection with $x=0$, and at points $x=\pm 1$, curvature is stationary (Fig. 34). Therefore it is natural to approximate this graph by two parabolas with apex/vertexes on the straight lines $x=\pm 1$, arrange/located it is symmetrical relative to the origin of coordinates, and by the smoothly adjacent them rectilinear cut (Fig. 35).

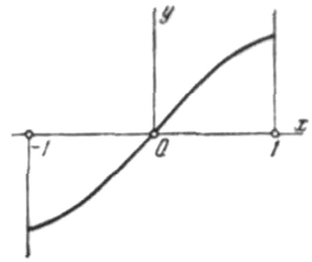

Fig. 34.

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Function y(x), given by this graph, we will characterize two parameters k and l. Value k - this is the angular coefficient of the inclined sections of graph y'(x), and l - value y'(x) in zero. Thus, k=|y''| with x=±1, and l=y'(0).

By simple examination it shows that the region of the allowed values of the parameter l, determined the condition

$$\int_{-1}^{1} y'^2\,dx = 1,$$

will be

$$\frac{1}{\sqrt{2}} \leqslant l \leqslant \sqrt{\frac{3}{2}}.$$

So it shows that the values λ, which characterize the common/general/total deformation of shell, are limited, precisely,

$$\lambda < \frac{4}{3},$$

$$\int_{-1}^{1} y'^2\,dx = 1.$$

Let us assume now

$$\frac{b}{a}=\xi, \quad h=\frac{1}{2}\left(\frac{a\pi}{2n}\right)^2\frac{\lambda}{b}.$$

Then we obtain

$$U=\frac{\pi^2 E\delta^3\lambda}{12(1-\nu^2)n\xi^3}\int_{-1}^{1} y''^2\,dx+\frac{\pi^2\nu E\delta^3\lambda}{6(1-\nu^2)n\xi}+$$
$$+2^{1/4}c\pi^3\frac{1}{4n^2}E\delta^{5/2}R^{1/2}\lambda^{1/4}\int_{-1}^{1}|y''|^{1/2}\left(1+\frac{2\lambda}{\xi^2}y'^2\right)dx+\text{const.}$$

As shown above, at the moment of the loss of stability of shell, we have

$$\xi=1, \quad n=0{,}91\sqrt{\frac{R}{\delta}}.$$

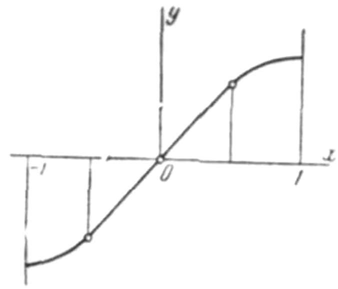

Fig. 35.

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Substituting these values in the expression of strain energy, we will obtain

$$U=\frac{\pi^2 E\delta^3}{12(1-\nu^2)n}\left(\lambda\int_{-1}^{1} y''^2\,dx+0{,}6\lambda+\right.$$
$$\left.+2\lambda^{1/4}\int_{-1}^{1}|y''|^{1/2}\left(1+2\lambda y'^2\right)dx\right)+\text{const}$$

It is here accepted $\nu=0{,}3$, c=0.19.

Let us assume

$$J(y)=\lambda\int_{-1}^{1} y''^2\,dx+0{,}6\lambda+2\lambda^{1/4}\int_{-1}^{1}|y''|^{1/2}\left(1+2\lambda y'^2\right)dx.$$

For functioning y(x) the form indicated we have

$$J=2kl\lambda+\frac{4l}{\sqrt{k}}\lambda^{1/4}+\frac{8l^3}{3\sqrt{k}}\lambda^{5/4}+0{,}6\lambda,$$

while the condition

$$\int_{-1}^{1} y'^2\,dx=1$$

transfer/converts in communication/connection between k and l

$$\frac{1}{k}=\frac{3}{2l}-\frac{3}{4l^3}.$$

For determining of minimum J and values k, l , at which this minimum is reached, were calculated values J for different l of the interval indicated above

$$\frac{1}{\sqrt{2}}\leqslant l\leqslant\sqrt{\frac{3}{2}}$$

and values λ, not exceeding 4/3. In this case, it turned out that minJ with λ=const is in practice always obtained with one and the same value $l \approx 0.82$. But since value J at the point where is reached the minimum, is stationary, then, without accomplishing large error, it is possible to consider that minimum J is equal to its value with $l=0.82$. Then we obtain

$$\min J=4,1\lambda+2,24\lambda^{1/4}+\lambda^{5/4}.$$

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Consequently, energy of elastic deformation of shell in the state of equilibrium is equal to

$$U=\frac{\pi^2E\delta^3}{12(1-\nu^2)n}\left(4,1\lambda+2,24\lambda^{1/4}+\lambda^{5/4}\right)+\text{const}.$$

Let us turn now to work A, produced by the external load p. We have

$$A=2\pi R\,\delta ph.$$

Introducing here instead of h parameter λ according to the equality

$$h=\frac{1}{2}\left(\frac{a\pi}{2n}\right)^2\frac{\lambda}{b},$$

and noting that

$$\frac{b}{a}=\xi=1, \quad a=\frac{\pi R}{n}, \quad n=0{,}91\sqrt{\frac{R}{\delta}}, \quad p=\bar{p}E\frac{\delta}{R},$$

let us have

$$A=\frac{\pi^4}{3{,}3n}E\delta^3\bar{p}\lambda.$$

For a shell, which is found in the state of elastic equilibrium, we have

$$d(U-A)=0.$$

Hence is obtained the value of the dimensionless load $\bar{p}$ depending on the parameter λ, which characterizes the axial compression

$$\bar{p}=0{,}03\times$$
$$\times(4{,}1+0{,}56\lambda^{-3/4}+1{,}25\lambda^{1/4}).$$

Let us recall that here as everywhere in the analogous cases, the parameter λ cannot be taken as to small ones as desired, since it characterizes the deformation which is assumed to be considerable. Graphically dependence of $\bar{p}$ on λ is represented in Fig. 36. We see that the received by shell load after loss of stability falls. Small value $\bar{p}\simeq0.18$.

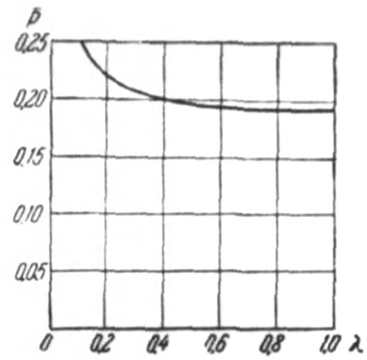

Fig. 36.

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Thus, the lower critical load p for a cylindrical shell during axial compression is determined from the formula

$$p = 0{,}18E\frac{\delta}{R}.$$

The obtained formula for p was subjected to experimental check. The experimental determination of critical loads during the axial compression of cylindrical shell was produced during the installation which is schematically depicted in Fig. 37. The basic cell/element of the installations are two strictly parallel disks 1 which with the tightening of nut 2, carried out in the form of steering control, converge and compress experience/tested shell 3.

Lower disk has a bushing, in which is passed the rod of upper

disk. Both of disks after the articulation of bushing and rod according to the sliding landing/fitting are processed from one installation, than is reached by strict parallelism to each other.

The tested shells of radius R=40 mm, length L=80 mm were obtained from copper by metal spraying in vacuum to the geometrically modern polished cylindrical surface. The edges of shell were trimmed on special mandrel/mount also from one installation of machine tool.

In order to ensure the complete uniformity of the distribution of compressive force according to the edge of shell, between supporting disks 1-1 and end/faces of shell, is placed fine/thin elastic it is placed fine/thin elastic packing to 4. The edges of shell were centered on conical washers by 8.

Compressive force was recorded with the aid of the strain gauges, mounted with ring 5, through which was transferred the effort/force from screw/propeller by 6 to the rod of upper supporting disk 1. Strain gauges were calibrated to the value of compressive force with the aid of removable load 7.

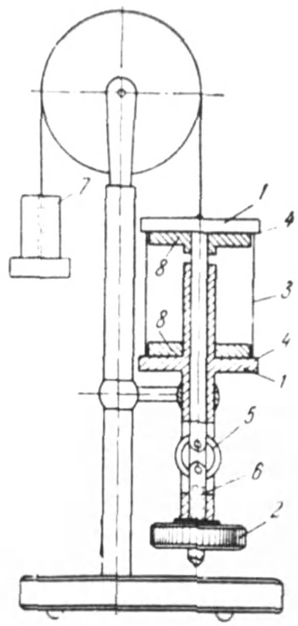

Fig. 37.

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Experiment was conducted as follows. Test specimen was installed between supporting disks 1-1. With the aid of removable load 7, were calibrated readings of strain gauges to the value of compressive force. Then by the rotation of steering control 2 shells was compressed. Compressive force was recorded by reading the galvanometer, connected to strain gauges. When compressive force reached upper critical value f_e. shell lost stability with the formation of the system of the correctly arranged/located dents on

its surface (Fig. 38).

As a result of loss of stability, the received by shell load descended and it continued to descend during further approach of supporting disks. Finally, it reached minimum value f_l (lower critical value).

Testing underwent the cylindrical shells of radius R=40 mm, height/altitude L=80 mm with different thickness δ from 0.03 to 0.09 mm. Figure 39 depicts the graph/diagrams of the theoretical dependence of values f_e and f_l from the thickness of the shell

$$f_e = 0{,}6E\frac{\delta}{R}(2\pi R\delta), \quad f_l = 0{,}18E\frac{\delta}{R}(2\pi R\delta).$$

The module/modulus of elasticity E for copper is accepted equal to $0.9 \cdot 10^6$ kg/cm². (This average/mean value of the modulus of elasticity for the copper specimen/samples, obtained by metal spraying in vacuum). The isolated points, designated by small circles, give the experimental value of value f_e. As is evident, these values are close to theoretical ones. One should, however, note that the nearness of f_e experimental value theoretical was obtained only after the careful adjustment of installation and preparation for the specimen/samples, ensuring the uniformity of the distribution of compressive force according to the edge of shell.

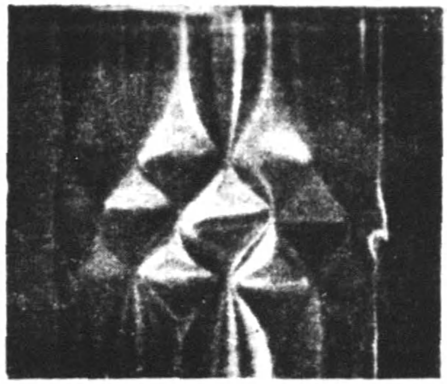

Fig. 38.

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Experimental study of the value of upper critical force f_e gives grounds to confirm that in real construction/design theoretical value f_e in practice never is reached, and at this value of critical force it cannot be been oriented during the calculation of shells to stability.

The experimental values of lower critical value in Fig. 39 are noted by dark circles. Lower critical value f_l was characterized by considerable stability. Its experimental value was close to theoretical in all cases, including when upper critical value f_e was much lower than the theoretical. Furthermore, value f_l did not virtually change during repeated tests, what cannot be said about

upper critical value.

The comparison of experimental value f_1 with its theoretical value gives grounds to recommend as computed value for critical load p of the compressed cylindrical shell the obtained above formula

$$p=0,18E\frac{\delta}{R}.$$

3. Effect of initial bending on stability. Limitedly elastic shells. Our all preceding/previous examinations were related to the unlimitedly elastic, geometrically modern shells. Real shell is limited by elastic, and its form far is not modern. Both these facts can influence the results, obtained in the preceding/previous point/item, in particular, to the value of critical loads.

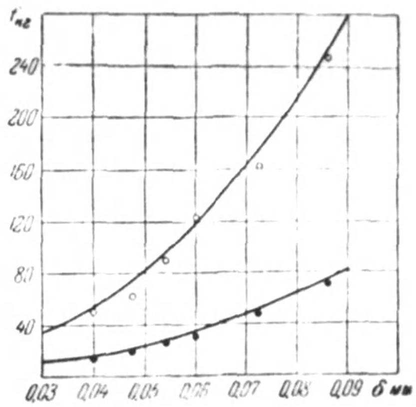

Fig. 39.

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Since the upper critical load is determined from the examination of comparatively small deformations of surface, adverse initial bending can considerably change it. Lower critical load, being it is connected with the large deformations of shell, is less sensitive to initial bending, but the limited elasticity of material can influence it significantly. Now we will examine the influence of the initial bending of shell on upper critical load and effect of the limited elasticity of material for lower critical load.

It is obvious, the effect of initial bending on the stability of shell will be greatest during to the assigned magnitude bending ε, if this bending reproduces the form of shell during supercritical

deformation. It is logical to consider that the critical load with which the shell with bending loses stability, descends to the value of the received by shell load during the appropriate supercritical deformation.

The received by shell load p during supercritical deformation is determined from formula (p. 2)

$$\bar{p}=0{,}03\,(4{,}1+0{,}56\lambda^{-3/4}+1{,}25\lambda^{1/4}).$$

where λ - the parameter, which characterizes deformation (axial compression). Let us establish communication/connection between axial compression (λ) and maximum normal sagging/deflection which let us designate ε. As shows the calculation, given in p. 2, the curve y(x), which assigns the shape of surface of shell during supercritical deformation, it is determined one and the same parameters l to k(l ≈0.82). Hence it follows that in initial the variables x, y will

$$\frac{1}{y_{max}^2}\int\limits_{(b)} y'^2\,dx=\text{const},$$

i.e., the to the left confronting value it does not depend on the amount of deformation. If we in this relationship/ratio pass to by the variable $\bar{x}$, $\bar{y}$ as this having done in the preceding/previous point/item, placed

$$x=\frac{b}{2}\bar{x},\quad y=\frac{2n}{\pi}\sqrt{bh}\,\bar{y},$$

that we will obtain

$$\frac{1}{y_{max}^2}\int\limits_{(b)} y'^2\,dx=\frac{2}{b\bar{y}_{max}^2}\int\limits_{-1}^{1}\bar{y}'^2\,d\bar{x}.$$

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For function $\bar{y}(\bar{x})$, determined by parameters k and l (para. 2), we have

$$\bar{y}_{\max} \simeq 0.65, \quad \int_{-1}^{1} \bar{y}'^2 \, d\bar{x} = 1.$$

The value of the maximum sagging/deflection of shell during the deformation in question is equal to

$$\varepsilon = \tilde{y}_{\max} = \frac{\pi}{2n} y_{\max}.$$

The axial compression of shell is equal

$$h = \frac{\pi^2}{8n^2} \int_{(b)} y'^2 \, dx.$$

We hence obtain the relationship/ratio between axial compression h and maximum transverse sagging/deflection

$$\frac{8n^2 h}{\pi^2} \left(\frac{\pi}{2n\varepsilon}\right)^2 = \frac{1}{(0{,}65)^2} \frac{2}{b}.$$

If we here introduce instead of h the parameter λ, determined on the formula

$$h = \frac{1}{2} \left(\frac{a\pi}{2n}\right)^2 \frac{\lambda}{b},$$

then we will obtain the following dependence between ε and λ:

$$\varepsilon = 1{,}6\lambda^{1/2} \sqrt{R\delta}.$$

If we substitute determined by this relationship/ratio value λ into formula for $\bar{p}(\lambda)$, then we will obtain the evaluation of the effect of initial bending on stability. Figure 40 this dependence

depicts graphically. Let us note that the graph is constructed for the comparatively large values of parameter $\varepsilon/\sqrt{R\delta}$. This is connected with the fact that our all examinations are related to supercritical deformations with considerable changes in exterior form of shell.

Let us examine now the effect of the limited elasticity of the material of shell on the value of lower critical load.

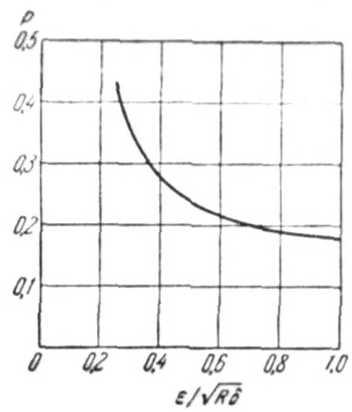

Fig. 40.

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The results of the investigation p. 2 are used only to such shells with the limited elasticity, for which the there supercritical deformations in question do not lead to the voltage/stresses, which emerge beyond elastic limit. In order to explain, what must be this shell, let us find maximum voltage/stresses during these deformations.

It is obvious, maximum voltage/stresses appear in the zone of the powerful local bending of shell along fin/edges. For these voltage/stresses σ, we have the formula

$$\sigma = c'E\left(\frac{1}{\rho}\right)^{1/2}\delta^{1/2}\alpha^{3/2}.$$

where ρ - a radius of curvature of fin/edge, α - an angle between the

plane of fin/edge and the tangential planes of the surface of shell along fin/edge, but c' - constant, equal to approximately 0.9.

Let us find α and ρ. We have

$$\alpha = \frac{\pi}{2n}.$$

In the initial variables x, y curvature in the apex/vertexes of fin/edges (it there greatest) is equal to

$$\frac{1}{\rho} = |y''|.$$

In the dimensionless variables χ, y we have

$$\frac{1}{\rho} = \frac{2n}{\pi} \sqrt{bh}\, \frac{4}{b^2} |y''|.$$

Taking into account, that

$$h = \frac{1}{2} \left(\frac{a\pi}{2n}\right)^2 \frac{\lambda}{b}, \quad b = a = \frac{\pi R}{n},$$

we will obtain

$$\frac{1}{\rho} = \frac{4n\sqrt{\lambda}\,|y''|}{\sqrt{2}\,\pi R}.$$

In our approach/approximation of function $y(\chi)$ (see Section 2) value $|y''| = k$. Therefore

$$\frac{1}{\rho} = \frac{4n\sqrt{\lambda}\,k}{\sqrt{2}\,\pi R}.$$

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Substituting the value $1/\rho$ and α in formula for σ and noting that

$$n = 0{,}91 \sqrt{\frac{R}{\delta}},$$

let us have

$$\sigma \simeq 2E \frac{\delta}{R} (k \sqrt{\lambda})^{1/2}.$$

Into this formula enter values k and λ. As far as value is concerned k, it is connected with l the relationship/ratio

$$\frac{1}{k} = \frac{3}{2l} - \frac{3}{4l^3}.$$

In the state of elastic equilibrium $l \simeq 0.82$. Hence for k is obtained value $\simeq 2$. Value λ changes in supercritical deformation. Its maximum value corresponds to transition to steady states of equilibrium after "cotton/knock", i.e., the deformation, during which the received by shell load is smallest (p_l). This value λ is approximately equal to 1.35 (see Section 2).

Substituting the value k=2 and λ=1.35 in formula for σ, let us find the maximum voltage/stresses in the material of shell during the supercritical deformations, caused by loss of stability:

$$\sigma = 3E \frac{\delta}{R}.$$

Hence we consist that the examination p. 2, the in particular obtained there formula for the lower critical load

$$p_l = 0{,}18E \frac{\delta}{R},$$

they are related only to such shells whose voltage/stresses by value 3Eδ/R do not cause noticeable plastic deformations. This condition can be considered carried out, if

$$3E \frac{\delta}{R} < \sigma_{e},$$

where σ_e - time/temporary strength of materials. Such shells we will

call fine/thin.

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Let us examine for an example the steel shells. Set/assuming

$$E = 2 \cdot 10^6 \quad \text{kg/cm}^2, \qquad \sigma_s = 4 \cdot 10^3 \quad \text{kg/cm}^2,$$

we will obtain

$$\frac{R}{\delta} > 1250.$$

This example shows that our examinations, until now, were related to very films. The practically important case R/δ=500-1250 render/showed out of this examination. In connection with this we will continue our investigation.

According to our representations, the supercritical deformation of shell is accompanied by the appearance of fin/edges on its surfaces which in deformation change their form. If local bending in fin/edge proves to be so considerable that on the surface of shell appear the plastic deformations, then, as shown in chapter 1, §2, deformation in this place stops. Supercritical deformation with the advent of the irreversible changes in the zone of powerful local bending proves to be energetically unfavorable. Therefore, minimizing strain energy U during assigned/prescribed axial compression (λ), we

must place as supplementary condition the limitation of the voltage/stresses in the zone of powerful local bending. This condition appears as follows:

$$(k\sqrt{\lambda})^{1/2} \leqslant 0.5\,\frac{\sigma_e}{E}\left(\frac{R}{\delta}\right),$$

where k and λ have **previous value (p. 2).** For the shells in which

$$3E\frac{\delta}{R} \geqslant \sigma_e,$$

during the deformations, **which corres**pond to lower critical load and close to them, under the condition indicated must occur the equality, i.e.,

$$(k\sqrt{\lambda})^{1/2} = 0.5\,\frac{\sigma_e}{E}\left(\frac{R}{\delta}\right). \qquad (*)$$

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In p. 2, we found the following expressions for energy of elastic deformation U and of work A, produced by the external load:

$$U = \frac{\pi^2 E\delta^3}{12(1-\nu^2)\,n} J + \text{const},$$

$$J = 2kl\lambda + \frac{4l}{\sqrt{k}}\lambda^{1/4} + \frac{8}{3}\frac{l^3}{\sqrt{k}}\lambda^{5/4} + 0.6\lambda,$$

$$A = \frac{\pi^4}{3.3n} E\delta^3 \bar{p}\lambda.$$

For the states of the equilibrium of shell under the load, close to lower critical, parameters k and l in reason indicated above are connected by relationship/ratio (*). Furthermore, we have relationship/ratio **between k and** l

$$\frac{1}{k} = \frac{3}{2l} - \frac{3}{4l^3}.$$

Thus, both of values U and A can be considered depending from one parameter.

As this parameter we will take

$$s=\frac{l}{k}.$$

If we introduce parameter s into the expression of strain energy U and by the produced external load of work A, then from the condition of the equilibrium of the shell

$$d(U-A)=0$$

is obtained the following expression for the received by shell load:

$$\bar{p}=0{,}03\left\{0{,}6+\frac{2\omega s^2+\left(0{,}5+\frac{1}{\omega^3}\right)}{s-s^2}\right\},$$

where

$$\omega=0{,}5\frac{\sigma_s}{E}\frac{R}{\delta}.$$

Minimizing expression $\bar{p}$ from parameter s, we find lower critical load in dependence on the elasto-plastic properties of material (parameter ω). Figure 41 depicts the graph/diagram of this dependence.

Let us explain the region of the applicability of the obtained dependence.

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For this let us, first of all, note that in view of condition $\sigma_s < 3E\frac{\delta}{R}$ we have

$$\omega = 0.5\frac{\sigma_s}{E}\frac{R}{\delta} < 1.5.$$

Thus, for values $\omega > 1.5$ it is necessary to count $\bar{p}=0.18$. Further, dependence $\bar{p}(\omega)$ cannot be used at too low values ω, since in this case the supercritical deformation, which corresponds to lower critical load, it is not possible to count considerable. Calculation shows that the supercritical deformations limitedly elastic shells with this value of the parameter ω are limited to the condition

$$\lambda \leqslant \frac{2}{3}\omega^4.$$

Therefore, if ω is noticeably less than unity, then λ is very small, and all our examinations they are related to such supercritical deformations which are accompanied by a considerable change in exterior form.

The obtained dependence of the lower critical value of load from the elasto-plastic properties of material according to character corresponds to the data of experimental studies.

4. Narrow cylindrical panels during axial compression. In order to strengthen cylindrical shell, worker under conditions for axial compression, her they support by the rigid longitudinal elements

along generatrices. In this case, the shell is divide/marked off into narrow cylindrical panels. Let us examine a question concerning supercritical deformations and critical loads for such panels.

Let the hinged attached on edges narrow cylindrical panel lose stability under the action of the axial compressive load p.

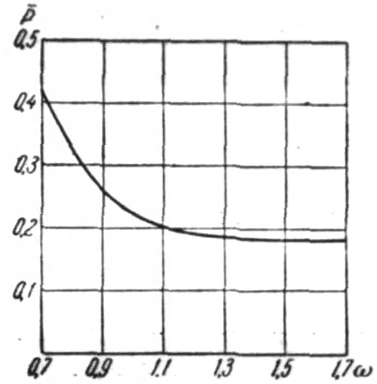

Fig. 41.

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Just as for the closed cylindrical shells, the transverse sagging/deflection w of shell satisfies the differential equation

$$\frac{P}{\delta}\Delta\Delta\Delta\Delta w+\frac{E}{R^2}\frac{\partial^4 w}{\partial x^4}+p\Delta\Delta\frac{\partial^2 w}{\partial x^2}=0.$$

It shows that solution w(x, y) of this equation under the condition for the hinged support of panel for edges takes the form

$$w=c\sin\frac{2\pi mx}{L}\sin\frac{ny}{R}.$$

In contrast to the case of the closed cylindrical shell where both of parameters m and n wholes, for a panel with a width of b parameter n is limited to the condition of the integrality of the expression

$$\frac{nb}{\pi R}.$$

Load p, calling the loss of stability of panel, is connected with parameters m and n by the relationship/ratio

$$\frac{D}{\delta}\left(\frac{4\pi^2m^2}{L^2}+\frac{n^2}{R^2}\right)^4+\\+\frac{E}{R^2}\left(\frac{2\pi m}{L}\right)^4-p\left(\frac{4\pi^2m^2}{L^2}+\frac{n^2}{R^2}\right)^2\left(\frac{2\pi m}{L}\right)^2=0.$$

which after the introduction of the new parameters

$$\xi=\frac{Ln}{2\pi Rm}, \qquad \eta=\frac{n^2\delta}{R}$$

and of the dimensionless load

$$\bar{p}=p\,\frac{R}{E\delta}$$

takes the form

$$\bar{p}=\frac{(1+\xi^2)^2\,\eta}{12\,(1-\nu^2)\,\xi^2}+\frac{\xi^2}{(1+\xi^2)^2\,\eta}.$$

Let us find the small value of the load, capable of causing the loss of stability of panel. Set/assuming

$$\zeta=\frac{(1+\xi^2)^2}{\xi^2}\,\eta,$$

let us have

$$\bar{p}=\frac{\zeta}{12\,(1-\nu^2)}+\frac{1}{\zeta}.$$

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In order to determine $\min \bar{p}$, it is necessary to, first of all, find the region of the allowed values of the parameter ζ. In connection with this let us note that from the integrality of expression nb/πR it follows

$$\frac{nb}{\pi R} \geqslant 1.$$

Therefore the parameter

$$\eta \geqslant \frac{\pi^2 R\delta}{b^2}.$$

Further, despite all values $\xi > 0$

$$\frac{\xi^2+1}{\xi} \geqslant 2.$$

Consequently,

$$\zeta = \frac{(1+\xi^2)^2}{\xi^2}\eta \geqslant \frac{4\pi^2 R\delta}{b^2}.$$

Now we will refine, that we bear in mind, speaking, that the panel is narrow. Let us call/name the panel of narrow, if its width

$$b < \frac{2\pi\sqrt{R\delta}}{\sqrt[4]{12(1-\nu^2)}}.$$

In view of this condition for the narrow panels

$$\zeta > \sqrt{12(1-\nu^2)}.$$

It is easy to see that the absolute minimum $\bar{p}$ is obtained with

$$\zeta = \sqrt{12(1-\nu^2)}.$$

But this means that for narrow panels it be reached cannot. In view of the monotonicity of increase $\bar{p}(\zeta)$ when $\zeta > \sqrt{12(1-\nu^2)}$, for narrow panels the minimum $\bar{p}(\zeta)$ is obtained at the smallest allowed value ζ. Let us find it.

Since

$$\frac{1+\xi^2}{\xi} \geqslant 2, \quad \eta \geqslant \frac{\pi^2 R\delta}{b^2},$$

the small ζ is equal

$$\zeta_0 = \frac{4\pi^2 R\delta}{b^2}.$$

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Consequently,

$$\bar{p}_e = \min \bar{p} = \frac{4\pi^2 R\delta}{12(1-\nu^2) b^2} + \frac{b^2}{4\pi^2 R\delta}.$$

This value p is obtained with

$$\xi = 1, \quad \eta = \frac{\pi^2 R\delta}{b^2}.$$

Thus, for narrow cylindrical panels upper critical load $\bar{p}_e$ is determined from the formula

$$\bar{p}_e = \frac{4\pi^2 R\delta}{12(1-\nu^2) b^2} + \frac{b^2}{4\pi^2 R\delta}.$$

Parameter n, which characterizes wave formation at the moment of loss of stability, is found from the relationship/ratio

$$\frac{nb}{\pi R} = 1.$$

Supercritical deformation is accompanied by the appearance of square bulges in one row/series along generatrix on entire width panel.

Let us characterize the narrowness of cylindrical panel the parameter τ,

$$\tau = \frac{(12(1-\nu^2))^{1/4}}{2\pi} \frac{b}{\sqrt{R\delta}}.$$

For the narrow panels

$$\tau < 1.$$

Upper critical load for a panel allow/assumes the following representation:

$$p = p_0 \frac{\tau^2 + \frac{1}{\tau^2}}{2},$$

where p_0 - upper critical load for the closed cylindrical shell during axial compression.

Let us find the lower critical load of narrow panel. The supercritical elastic state of cylindrical panel during axial compression we identify with the supercritical state of closed shell at the condition of equality parameters m and n, which characterize these states.

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After making this assumption, we can use the finished, obtained in p. 2 expression for the strain energy U and of produced by load work A [1]).

FOOTNOTE [1]. It goes without saying, in this case, are disregarded the conditions of attachment for by generatrix panels. Therefore the value of the lower critical load to which we come, will be less than the true. However, there is the foundations for considering that it all the same is close to it at least for not too narrow panels. ENDFOOTNOTE.

We have (page 188)

$$U=\frac{\pi^2E\delta^3\lambda}{12(1-\nu^2)n\xi^3}\int_{-1}^{1}y''^2\,dx+\frac{\pi^2\nu E\delta^3\lambda}{6(1-\nu^2)n\xi}+$$

$$+\frac{2^{1/4}c}{4n^2}\pi^3E\delta^{5/2}R^{1/2}\lambda^{1/4}\int_{-1}^{1}|y''|^{1/2}\left(1+\frac{2\lambda}{\xi^2}y'^2\right)dx+\text{const.}$$

For a panel $\xi=1,$ while n is determined from the relationship/ratio

$$\frac{nb}{\pi R}=1,$$

where b - width of panel. If we introduce these values ξ and n into expression for U, then it takes the following form:

$$U=\frac{\pi^2E\delta^3}{12(1-\nu^2)n}\left\{\lambda\int_{-1}^{1}y''^2\,dx+2\nu\lambda+\right.$$

$$\left.+2\tau\lambda^{1/4}\int_{-1}^{1}|y''|^{1/2}(1+2\lambda y'^2)\,dx\right\}+\text{const}$$

Let us find the produced by load work. We have (page 190)

$$A = 2\pi R\delta ph,$$

$$h = \frac{1}{2}\left(\frac{\pi}{2n}\frac{\pi R}{n}\right)^2 \frac{n}{\pi R}\lambda,$$

$$n = \frac{\pi R}{b}, \quad b^2 = \frac{4\pi^2 R\delta\tau^2}{\sqrt{12(1-\nu^2)}}.$$

With the aid of these relationship/ratios the expression for work A is reduced to the form

$$A = \frac{\pi^4}{3{,}3n} E\delta^3 \bar{p}\lambda\tau^2.$$

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Just as in the case of closed shells, true shape of surface and, consequently, also strain energy, we determine from the condition of the minimum of functional U(y) during the assigned/prescribed axial compression. This is equivalent to task to the minimum for a functional

$$J_\tau = \lambda \int_{-1}^{1} y''^2\,dx + 0{,}6\lambda + 2\tau\lambda^{1/2} \int_{-1}^{1} |y''|^{3/2}(1 + 2\lambda y'^2)\,dx$$

under conditions

$$\int_{-1}^{1} y'^2\,dx = 1,$$

$$y'(-1) = y'(1) = 0.$$

With τ=0 this task has the obvious solution

$$y_0(x) = \frac{2}{\pi} \sin \frac{\pi x}{2}.$$

Let the function $y_\tau(x)$ realize the minimum of functional J_τ with small τ. It is obvious, it is close to $y_0(x)$. In view of the stability of functional during the function, which realizes the minimum, it is possible to take as $\min J_\tau$ equal to its value during function $y_0(x)$. Permissible in this case error will be of the order τ^2.

Substituting in the expression of functional J_τ function $y_0(x)$, we will obtain

$$J_\tau = \left(\frac{\pi^2}{4} + 0{,}6\right)\lambda + \tau(3{,}8\lambda^{1/4} + 3\lambda^{5/4}).$$

Now from the condition of the equilibrium of the shell

$$\frac{d}{d\lambda}(U - A) = 0$$

we find the received by shell load p in dependence on deformation (λ)

$$\bar{p} = \frac{0{,}03}{\tau^2}\left(3{,}1 + \tau\left(0{,}95\lambda^{-3/4} + 3{,}75\lambda^{1/4}\right)\right).$$

$$\min \bar{p} \simeq \frac{0{,}03}{\tau^2}(3{,}1 + 4{,}6\tau).$$

The investigation of a question concerning critical loads for narrow panels we is summed up by following conclusion.

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For the narrow cylindrical panel, hinged along sides, during axial compression upper critical load is equal to

$$p_e = \bar{p}_e E \frac{\delta}{R}, \quad \bar{p}_e = 0{,}3\left(\tau^2 + \frac{1}{\tau^2}\right);$$

lower critical load is equal to

$$p_l = \bar{p}_l E \frac{\delta}{R}, \quad \bar{p}_l = \frac{0{,}03}{\tau^2}(3{,}1 + 4{,}6\tau).$$

The parameter τ through the width of panel b, radius of curvature R and thickness δ is determined by the equality

$$\tau = \frac{(12(1-\nu^2))^{1/4}}{2\pi} \frac{b}{\sqrt{R\delta}}.$$

Panel is considered narrow, if $\tau<1$.

5. Structurally orthotropic cylindrical shells during axial compression. The cylindrical shell, reinforced by elastic stringers, with a sufficient deneness of the arrangement of the latter can be considered as orthotropic. In present point/item we will examine this shell, which is found under conditions for axial compression.

Thus, let the circular cylindrical shell of radius R, of length L and of thickness δ be supported by the densely arranged stringers. Let f - sectional area of stringer, I - moment of inertia in reference

plane (radial plane), and l — distance between stringers. The modulus of elasticity of sheathing/skin and stringers let us designate through E, but lateral contraction ν for simplicity of unpacking/facings let us place equal to zero.

Let us introduce on the surface of shell the orthogonal coordinates x, y, after accepting for lines x linear generator. The module/moduli of elasticity E_1 and E_2 according to directions x and y of shell as orthotropic will be

$$E_1 = E\left(1 + \frac{\delta'}{\delta}\right), \quad E_2 = E,$$

where δ' - total sectional area of stringers per the unit of length along lines y, i.e.

$$\delta' = \frac{f}{l}.$$

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The flexural rigidity of shell in the plane, perpendicular to stringers, is equal to

$$D_2 = D = \frac{E\delta^3}{12}.$$

Flexural rigidity in reference plane is equal to

$$D_1 = \frac{E\delta^3}{12} + \frac{EI}{l}.$$

Fundamental equations for an orthotropic shell, which is found under conditions for axial compression, take the following form:

$$\frac{D}{\delta}\Omega w = -p\frac{\partial^2 w}{\partial x^2} + \frac{1}{R}\frac{\partial^2 \Phi}{\partial x^2},$$
$$\frac{1}{E}\Omega' w = -\frac{1}{R}\frac{\partial^2 w}{\partial x^2}.$$

Here w - the transverse sagging/deflection of shell at the moment of loss of stability under load p, Φ is the function of voltage/stresses while Ω and Ω' — differential operators,

$$\Omega = (1+\omega)\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4},$$

$$\Omega' = (1+\omega')\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4},$$

$$\omega = \frac{D_1 - D_2}{D_2} = \frac{12I}{\delta^3 l},$$

$$\omega' = E\left(\frac{1}{E_1} - \frac{1}{E_2}\right) = \frac{\delta'}{\delta+\delta'}.$$

If we from the equations of loss of stability exclude the function of voltage/stresses Φ, that for w is obtained the following equation:

$$\frac{D}{\delta}\Omega\Omega' w + p\frac{\partial^2}{\partial x^2}(\Omega' w) + \frac{E}{R^2}\frac{\partial^4 w}{\partial x^4} = 0.$$

Let us assume the form of shell at the moment of loss of stability in the form

$$w = c\sin\frac{2\pi m x}{L}\sin\frac{ny}{R}.$$

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If we substitute this expression w into equation for sagging/deflection, then we will obtain certain relationship/ratio for parameters m and n. Set/assuming for the brevity

$$\xi = \frac{Ln}{2\pi R m},\ \eta = \frac{n^2\delta}{R},\ \bar{p} = \frac{pR}{E\delta},$$

to this relationship/ratio it is possible to give the form

$$\bar{p}=\frac{\omega+(1+\xi^2)^2}{12\xi^2}\eta+\frac{\xi^2}{\omega'+(1+\xi^2)^2}\frac{1}{\eta}.$$

Let us find the smallest load, capable of causing the loss of stability of shell, i.e., upper critical load. With that fix/recorded ξ the minimum $\bar{p}(\xi,\eta)$ according to variable η is obtained, when

$$\frac{\omega+(1+\xi^2)^2}{12\xi^2}\eta=\frac{\xi^2}{\omega'+(1+\xi^2)^2}\frac{1}{\eta},$$

and this minimum is equal to

$$\bar{p}(\xi)=\frac{1}{\sqrt{3}}\left(\frac{\omega+(1+\xi^2)^2}{\omega'+(1+\xi^2)^2}\right)^{1/2}.$$

Let us assume that $\omega>\omega'$. The physical sense of this condition lies in the fact that the reinforcement of shell by stringers relatively more increases flexural rigidity, than to elongation - compression. Under this hypothesis

$$\min_{(\xi)}\bar{p}(\xi)=\frac{1}{\sqrt{3}}$$

and it is reached when $\xi=\infty$. But this value ξ is inadmissible, since

$$\xi=\frac{Ln}{2\pi Rm}.$$

while n and 2m take only integer values. In connection with this, and taking even into consideration the monotonicity of decrease $\bar{p}(\xi)$ with increase ξ. we let us take for m the small possible value m=0.5.

After accepting m=0.5 let us have

$$\frac{\xi^2}{\eta}=\frac{L^2}{\pi^2 R\delta}.$$

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Now $\bar{p}$ it is possible to present in the following form:

$$\bar{p}=\frac{\varepsilon(\omega-\omega')}{12}+\frac{\varepsilon}{12}\vartheta(\xi)+\frac{1}{\varepsilon\vartheta(\xi)},$$

where

$$\varepsilon=\frac{\pi^2 R\delta}{L^2},\quad \vartheta(\xi)=\omega'+(1+\xi^2)^2.$$

Minimum of $\bar{p}$ is reached at the condition

$$\frac{1}{12}\varepsilon\vartheta(\xi)=\frac{1}{\varepsilon\vartheta(\xi)} \qquad (*)$$

and it is equal to

$$\bar{p}_e=\frac{\pi^2(\omega-\omega')}{12}\frac{R\delta}{L^2}+\frac{1}{\sqrt{3}}.$$

If one assumes that value ξ, at which is reached min $\bar{p}$, is considerable, then it is possible to count

$$\vartheta(\xi)\simeq\xi^4.$$

Then relationship/ratio (*) for ξ takes the form

$$\frac{1}{12}\varepsilon\xi^4=\frac{1}{\varepsilon\xi^4}.$$

We hence find that ξ, while with it and parameter n:

$$\xi=12^{1/4}\frac{1}{\varepsilon^{1/4}},$$

$$n=\frac{\pi R}{L}\xi=(12\pi^4)^{1/4}\frac{R}{L}\left(\frac{L^2}{R\delta}\right)^{1/4}.$$

Let us note that $R\delta/L^2$ is usually small. Therefore our assumption relative to value ξ is not deprived of basis/bases.

If the setting of stringers considerably increases flexural rigidity, and not rigidity to elongation - compression, i.e., $\omega\gg\omega'$, then it is possible to count $\omega-\omega'\simeq\omega$. In this case, for an upper critical load, it is possible to accept the expression

$$\bar{p}_e=\frac{\pi^2\omega}{12}\frac{R\delta}{L^2}+\frac{1}{\sqrt{3}}.$$

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In order to present the visually obtained result, let us find critical force per the unit of the length of the section/cut of shell. We have

$$p_e\delta=\bar{p}_eE\frac{\delta}{R}\delta,$$

$$\frac{1}{\sqrt{3}}E\frac{\delta^2}{R}\simeq0.6E\frac{\delta}{R}\delta,$$

$$\frac{\pi^2\omega}{12}\frac{R\delta}{L^2}E\frac{\delta}{R}\delta=\pi^2\left(\frac{EI}{L^2}\right)\frac{1}{l}.$$

Value

$$0{,}6E\,\frac{\delta}{R}\,\delta$$

is critical force per the unit of the length of the cross section of the nonreinforced sheathing/skin, and

$$\pi^2\left(\frac{EI}{L^2}\right)\frac{1}{l}$$

there is Eulerian force for the isolated/insulated stringers. Thus, critical load for the shell in question consists of the critical load of the nonreinforced shell and critical load for the isolated/insulated stringers.

Let us examine now the supercritical deformations of stiffened shell. In connection with this let us find, first of all, expression for strain energy along fin/edges. In chapter 1, §1 (page 30) for an isotropic shell this energy per the unit of the length of fin/edge was calculated from the formula

$$\overline{U}_\gamma = \frac{D}{2}\int\limits_{-\bar{\varepsilon}}^{\bar{\varepsilon}} v''^2\,ds + \frac{\overline{D}}{2}\int\limits_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \frac{u^2}{\rho^2}\,ds + D\alpha\,(-2k + k_e + k_i).$$

For an anisotropic shell is obtained accurately the same expression. Only now D is flexural rigidity in the plane, perpendicular to

fin/edge, and $\bar{D}$ - rigidity to elongation - compression along fin/edge.

Let us assume

$$\omega = \frac{12D}{\delta^2 \bar{D}}, \quad \bar{\omega} = \frac{2\bar{D}}{E\delta}.$$

Then expression $\bar{U}_\gamma$ will take the form

$$\bar{U} = \frac{\delta E \bar{\omega}}{2} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \left(\omega \frac{\delta^2 v''^2}{12} + \frac{u^2}{\rho^2}\right) ds + \ldots,$$

where omitted add/composed

$$D\alpha(-2k + k_e + k_l).$$

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If from this point on, we repeat verbatim all reasonings which in the analogous case of isotropic shells are given in chapter 1, §1, then for strain energy we will obtain the expression

$$\bar{U}_\gamma = \frac{\bar{\omega}}{12^{5/4}} E\delta^{5/2}\alpha^{5/2}\rho^{-1/2} J_0 + \ldots$$

Here J_0 - minimum of the functional

$$J = \int_0^\infty (\omega v'^2 + u^2)\, ds$$

in the nonholonomic constraint

$$u' + v + \frac{v^2}{2} = 0 \qquad (**)$$

and under the boundary conditions

$$u(0)=0, \quad v(0)=-1, \quad u(\infty)=v(\infty)=0. \qquad (***)$$

In order to find minJ, it is expedient to introduce instead of the variables u and s new variables λu and λs. Relative to new variable coupling (**) and boundary conditions (***) are retained, but functional takes the form

$$J=\int_0^\infty\left(\frac{\omega v'^2}{\lambda^2}+\lambda^2 u^2\right)\lambda\,ds$$

(feature above new variables it is lowered). Subordinating now the indefinite factor λ to the condition

$$\lambda^4=\omega,$$

let us have

$$J=\omega^{3/4}\int_0^\infty(v'^2+u^2)\,ds.$$

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Thus, the determination of coefficient J_0 is brought to the determination of the minimum of the functional

$$\int_0^\infty(v'^2+u^2)\,ds$$

in communication/connection (**) and under boundary conditions (***). In chapter 1, §1, this minimum is found, and for it obtained value

~~1.15.

Consequently,

$$J_0 = 1.15\omega^{3/4}.$$

As shown above, the loss of stability of the circular cylindrical shell, reinforced by stringers, is accompanied by the education/formation of continuous bulges all over length of shell ($m=0.5$), evenly distributed in circumference. We will assume that the supercritical deformation has the same character.

For energy U of the supercritical deformation of isotropic shell, we have a formula (page 188)

$$U = \frac{\pi^2 E\delta^3\lambda}{12(1-\nu^2)n\xi^3}\int_{-1}^{1} y''^2\,dx + \frac{\pi^2(\nu-1)E\delta^3\lambda}{6(1-\nu^2)n\xi} +$$

$$+ \frac{2^{1/4}c}{4n^2}\pi^3 E\delta^{5/2}R^{1/2}\lambda^{1/4}\int_{-1}^{1}|y''|^{1/2}\left(1+\frac{2\lambda y'^2}{\xi^2}\right)dx + \text{const}.$$

Here first term calculates bending on the basis of the basic surface of shell. This bending occurs in radial planes, i.e., in the direction linear generator. Second term considers bending in the plane, perpendicular to fin/edge. In view of the fact that the regions of bulge go through entire length of shell, it is possible to count that this bending in the plane, perpendicular to generatrices. Third term/component/addend considers energy of complex deformation along fin/edge.

It is not difficult to consider, that if we use the same method of examination, also, for an orthotropic shell, then we will obtain analogous formula. It will differ from given by some factors before term/component/addends indicated due to change rigidity of shell. Specifically, first term to obtain the factor

$$D\Big/\frac{E\delta^3}{12},$$

where D - the flexural rigidity of orthotropic shell (reinforced by stringers), and an $E\delta^3/12$-rigidity of the nonreinforced shell. Second term will remain without change, and the third obtains the factor

$$\overline{\omega}\omega^{3/4}.$$

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As shown above, the flexural rigidity of stiffened shell in radial planes is equal to

$$D=\frac{E\delta^3}{12}+\frac{EI}{l},$$

where δ - thickness of shell, while EI/l - rigidity of the supporting stringers per the unit of the length of the cross section of shell. Thus,

$$D\Big/\frac{E\delta^3}{12}=1+\frac{12I}{l\delta^3}.$$

Let us turn to the second factor $\overline{\omega}\omega^{3/4}$. Examining this factor, let

us note that the parameter ξ is sufficiently great, and therefore the direction of fin/edges on the surface of shell is close to the direction of generatrices. Therefore value $\bar{\omega}$, equal to the ratio/relation of the rigidity to elongation - compression of the reinforced by stringers and nonreinforced shell, can be considered equal to $(\delta+\delta')/\delta$, where δ' - total area of stringers per the unit of the length of cross section. Estimating value ω, let us note that in view of the nearness of the direction of fin/edge to the direction of stringer, it is possible to consider flexural rigidity in the direction, perpendicular to fin/edge, equal to Eδ³/12. Therefore

$$\omega=\frac{\delta}{\delta+\delta'}.$$

Consequently, the factor

$$\bar{\omega}\omega^{1/4}=\left(\frac{\delta+\delta'}{\delta}\right)^{3/4}.$$

Let us assume for the brevity

$$D\Big/\frac{E\delta^3}{12}=\vartheta,\quad \bar{\omega}\omega^{1/4}=\vartheta'.$$

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Then the strain energy of orthotropic shell is written in the form

$$U=\frac{\vartheta\pi^2E\delta^3\lambda}{12n\xi^3}\int_{-1}^{1}y''^2\,dx+$$
$$+\vartheta'\,\frac{2^{1/4}c}{4n^2}\,\pi^3E\delta^{5/2}R^{1/2}\lambda^{1/4}\int_{-1}^{1}|y''|^{1/2}\left(1+\frac{2\lambda y'^2}{\xi^2}\right)dx+\text{const}.$$

In view of the fact that ξ greatly, it is possible to count

$$1+\frac{2\lambda}{\xi^2}y'^2\simeq 1.$$

Let us introduce into formula for U of the value

$$\xi=\left(\frac{12}{\pi^4}\right)^{1/8}\left(\frac{L^2}{R\delta}\right)^{1/4},$$

$$n=(12\pi^4)^{1/8}\left(\frac{L^2}{R\delta}\right)^{1/4}\frac{R}{L}.$$

Then we obtain

$$U=\frac{\pi^2E\delta^3}{12}\left(\frac{L}{R}\right)\left\{\frac{\pi}{\sqrt{12}}\left(\frac{R\delta}{L^2}\right)\vartheta\lambda\int\limits_{-1}^{1}y''^2\,dx+\right.$$

$$\left.+0{,}35\vartheta'\lambda^{1/4}\int\limits_{-1}^{1}|y''|^{1/2}\,dx\right\}+\text{const}.$$

Energy of the supercritical deformation of orthotropic shell in the state of elastic equilibrium is determined from stability condition of functional U(y) on function $y(x)$ with

$$\int\limits_{-1}^{1}y'^2\,dx=1.$$

This variational problem we will examine under the condition

$$\frac{\pi}{\sqrt{12}}\left(\frac{R\delta}{L^2}\right)\vartheta\gg 0{,}35\vartheta'.$$

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The physical sense of this condition lies in the fact that the reinforcement of shell by stringers very considerably strengthens the flexural rigidity of shell in radial planes. This assumption logically, since by the setting of stringers usually they want to achieve precisely this.

Under the condition indicated it is logical to assume that function $y(x)$, imparting to functional U(y) steady-state value, is close to function $y(x)$, that realizes the minimum of the functional

$$U' = \frac{\pi^2 E\delta^3}{12}\left(\frac{L}{R}\right)\left\{\frac{\pi}{\sqrt{12}}\left(\frac{R\delta}{L^2}\right)\vartheta\lambda\int_{-1}^{1} y''^2\,dx\right\} + \text{const},$$

which is obtained from functional U by discarding of subordinate in value term/component/addend. We will proceed from this assumption.

The function, which minimizes functional U' (y) under the condition

$$\int_{-1}^{1} y'^2\,dx = 1,$$

is equal to zero at the ends of interval (-1.1), it will be

$$y(x)=\frac{2}{\pi}\sin\frac{\pi x}{2}.$$

In view of the adopted assumption, the strain energy of shell in the state of elastic equilibrium we will obtain, if we substitute function $(2/\pi)\sin(\pi x/2)$ into formula for U:

$$U=\frac{\pi^2 E\delta^3}{12}\left(\frac{L}{R}\right)\left\{\frac{\pi^3}{4\sqrt{12}}\left(\frac{R\delta}{L^2}\right)\vartheta\lambda+0{,}68\vartheta'\lambda^{1/4}\right\}+\text{const.}$$

Let us find now work A, produced by axial compression. Just as in the case of isotropic shell, we obtain

$$A=\frac{\pi^4 ER\,\delta^2\lambda\bar{p}}{4n^3\xi}.$$

Substituting here the values of parameters n and ξ, let us have

$$A=\frac{\pi^3 E\delta^3}{4\sqrt{12}}\left(\frac{L}{R}\right)\lambda\bar{p}.$$

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From the condition of equilibrium, we now find

$$\frac{dU}{d\lambda}=\frac{dA}{d\lambda}$$

communication/connection between the deformation of shell (λ) and the received by shell load ($\bar{p}$). Specifically,

$$\bar{p}=\frac{\pi^2}{12}\left(\frac{R\delta}{L^2}\right)\vartheta+0{,}062\lambda^{-3/4}\vartheta'.$$

From formula for $\bar{p}$, we see that the received by shell load is

decreased during an increase in deformation (λ). The smallest value $\bar{p}$ is obtained at the maximum value λ, equal to $\simeq\pi^2/8$. Substituting this value in formula for $\bar{p}$, we find the value of the lower critical load

$$\bar{p}_l = \frac{\pi^2}{12}\left(\frac{R\delta}{L^2}\right)\vartheta + 0{,}053\vartheta'.$$

Since by hypothesis the setting of stringers considerably strengthens the flexural rigidity of shell in radial planes, then

$$\vartheta = 1 + \frac{12I}{l\delta^3} \simeq \frac{12I}{l\delta^3} = \omega.$$

$$\bar{p}_l = \frac{\pi^2\omega}{12}\left(\frac{R\delta}{L^2}\right) + 0{,}053\vartheta'.$$

Comparing this value with upper critical load, we consist that with sufficient rigidity of the supporting stringers the received by shell load after loss of stability is not in effect decreased, remaining equal to the approximately upper critical value

$$p_e = \frac{\pi^2\omega}{12}\left(\frac{R\delta}{L^2}\right) + \frac{1}{\sqrt{3}}.$$

§2. Cylindrical shells under external pressure.

In this paragraph we will examine the supercritical deformations of cylindrical shells appearing as a result of loss of stability under the evenly distributed external pressure. Just as in the case of axial compression (§1), this examination will be based on the application/use of principle A.

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1. Expression for functional W=U-A. Experiment shows that the loss of stability of cylindrical shell under external pressure is accompanied by the education/formation of continuous dents to entire length of shell with their regular arrangement in circumference. In connection with this functional W can be examined during isometric transformations of the type Z which are constructed in p. 1 §1. Let us recall these transformations.

Let us take right prism with a number of side faces 2n and will conduct on its any face α arbitrary curve γ, which connects the middles of the basis/bases of this face and which is unambiguously design/projected for the axle/axis of prism. It is reflected curve γ mirror in the radial plane, passing through the axle/axis of prism and the fin/edge of face α. In this case, we will obtain curve γ' in the face α', adjacent α. After this with the aid of face α' and curve γ' in it, we plot a curve γ'' in the face α'', adjacent α', and so forth. Thus, in each face α^i will be obtained curve γ^i. Let us conduct now arbitrary plane σ, perpendicular to the axle/axis of prism. It will cross curves γ^i at points A^i. Let P_σ – polygon with apex/vertexes A^i. When plane σ in parallel is displaced, polygon P_σ describes

isometric to cylinder surface Z. Curves γ' are fin/edges on this surface (Fig. 42).

Let us determine some geometric values for surface of Z. In connection with this in face α, let us introduce the rectangular Cartesian coordinates $x, y,$ after accepting the center of face in the origin of coordinates, and the straight lines, passing through the center of face in parallel to its sides - for coordinate axes. In these coordinates curve γ is assigned by the equation

$$y = y(x).$$

Curvature to curve γ is equal to

$$k = \frac{|y''|}{(1 + y'^2)^{3/2}}.$$

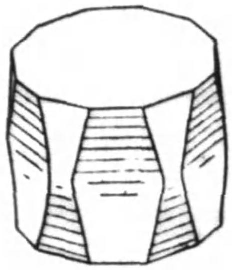

Fig. 42.

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If n is sufficiently great, or, it is better to say, face α is noticeably elongated in the direction of the axle/axis of prism, then for us interesting curves γ can be counted $1+y'^2 \simeq 1$ and therefore

$$k \simeq |y''|.$$

For an angle ϑ between the plane of fin/edge γ and the tangential planes of surface Z, is obtained the expression

$$\vartheta = \frac{\pi}{2n}\sqrt{1+y'^2}.$$

Here also it is possible to disregard y'^2 under radical sign. Then for an angle ϑ is obtained the expression

$$\vartheta \simeq \frac{\pi}{2n}.$$

During the determination of energy of elastic deformation of shell in the form of Z by us will be necessary the normal curvature $\tilde{k}$ of this surface. For it is obtained the expression

$$\tilde{k} = \frac{\pi}{2n} y''.$$

This is expression, just as preceding/previous, it is obtained

after analogous simplifications.

The produced by external pressure work is measured by the product of the value of pressure on a change in the volume, limited by shell, during deformation by the latter. In connection with this let us find the volume which limits surface of Z together with quadratic prisms which it close.

Intersection of surface of Z with plane σ, its perpendicular axes, is a 2n—angle plate P_σ. If we its area designate S, then the which interests us volume is equal to

$$V=\int_{-L/2}^{L/2} S\,dx.$$

Thus, in order to find volume V in dependence on function $y(x)$. that assigns shape of surface Z, we should, first of all, find expression S in dependence on y.

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Let a 2n—angle plate with apex/vertexes in the middles of the sides of the polygon, which lies at quadratic prism, be designated through P_0. Then independently of the position of plane σ in each 2n-angle plate P_σ of side they are parallel to sides P_0.

Plane σ intersects prism on the right 2n—angle plate $\bar{P}$. Let us number the apex/vertexes of this polygon in the order of their sequence and will designate through P' correct n—angle plate with the even apex/vertexes $\bar{P}$, but through P" - correct n—angle plate with the odd apex/vertexes $\bar{P}$. Polygon P_σ is entered in $\bar{P}$, and its sides form equal angles with sides $\bar{P}$. Hence it follows that polygon P_σ is obtained by the linear mixing of polygons P' and P" according to Minkowski[1].

FOOTNOTE [1]. By the linear combination of figures F' and F", which lie at one plane or parallel planes, is called the figure

$$F = \lambda F' + \mu F'',$$

which describes the terminus of the vector $r = \lambda r' + \mu r''$, when the terminuses of vectors r' and r" independently describe figures F' and F". See A. D. Aleksandrov, Convex polygons, State Technical Press, 1950. ENDFOOTNOTE.

Specifically,

$$P_\sigma = \frac{\frac{a}{2} - y}{a} P' + \frac{\frac{a}{2} + y}{a} P'',$$

where a - side of polygon $\bar{P}$.

On known formula the area of the polygon

$$P = \lambda P' + \mu P'' \quad (\lambda + \mu = 1)$$

is equal to

$$S = \lambda^2 S' + 2\lambda\mu S_{12} + \mu^2 S'', \quad S_{12} = 2S_0 - \frac{S' + S''}{2},$$

where S' and S" - areas of polygons P' and P" respectively, but S_0 - area of polygon P which is obtained in the linear combination P' and P" with $\lambda=\mu=1/2$. In our case S_0, it is the area of 2n-~~gon~~ angle P_0.

Taking into account, that $\lambda+\mu=1$ and S'=S", the formula for S can be converted. Specifically,

$$S = S' + 4\lambda\mu (S_0 - S').$$

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Substituting here

$$\lambda = \frac{\frac{a}{2} - y}{a}, \quad \mu = \frac{\frac{a}{2} + y}{a},$$

we will obtain

$$S = (*) - \frac{4y^2}{a^2} (S_0 - S'),$$

where $(*)$ - term/component/addend, not depending on y.

Introducing now into formula for volume V the obtained expression S, we will obtain

$$V = (*) - \frac{4}{a^2} (S_0 - S') \int_{-L/2}^{L/2} y^2 \, dx.$$

In connection with the forthcoming unpacking/facings it is

expedient to pass from the variables x, y, which we, until now, used, to new, dimensionless $\bar{x}, \bar{y}$, set/assuming

$$x=\frac{L}{2}\bar{x}, \quad y=\frac{a}{2}\bar{y}.$$

Here L - height/altitude of prism which we will identify with the height/altitude of initial shell, a - side of quadratic prism. If a radius of initial shell is equal to R, then

$$a=\frac{\pi R}{n}.$$

Interval of the variation of variables $\bar{x}$ and $\bar{y}$, obviously, is ~~to eat~~ $(+1, -1)$. In the new variables, the feature above designations of which subsequently let us lower, the which interest us geometric values will be calculated from the formulas:

$$k=\frac{2\pi R}{nL^2}|y''|,$$

$$\tilde{k}=\frac{\pi^2 R}{n^2 L^2}y'',$$

$$V=(*)-\frac{\pi^3 R^2 L}{8n^2}\int_{-1}^{1} y^2\,dx.$$

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Just as in the case of axial compression, for the strain energy of shell is obtained the following expression:

$$U=\frac{\pi^5 E\delta^3}{24(1-\nu^2)n^4}\left(\frac{R}{L}\right)^3\int_{-1}^{1} y''^2\,dx+$$

$$+\frac{\pi^3\nu E\delta^3}{12(1-\nu^2)n^2}\left(\frac{R}{L}\right)\int_{-1}^{1} y'^2\,dx+$$

$$+\frac{\pi^3 c}{4n^2}E\delta^{5/2}R^{1/2}\int_{-1}^{1}|y''|^{1/2}\,dx+\mathrm{const}.$$

Containing in this formula integral parameter n we will determine from periodicity condition of sagging/deflections for the circumference of shell at the moment of loss of stability.

The transverse sagging/deflection w of cylindrical shell at the moment of loss of stability under pressure q satisfies the differential equation

$$\frac{D}{\delta}\Delta\Delta\Delta\Delta w+\frac{E}{R^2}\frac{\partial^4 w}{\partial x^4}+\frac{qR}{\delta}\Delta\Delta\frac{\partial^2 w}{\partial y^2}=0. \qquad (*)$$

It shows that under the condition for the hinged support of shell the solution of this equation takes the form

$$w=c\sin\frac{\pi mx}{L}\sin\frac{ny}{R}.$$

Substituting this expression w in equation (*), we will obtain certain relationship/ratio between parameters m, n and by load q. If we instead of m and n introduce the parameters

$$\xi=\frac{\pi mR}{Ln},\quad \eta=\frac{n^2\delta}{R}$$

and the dimensionless load

$$\bar{q}=\frac{qR^3}{E\delta^3},$$

then to the relationship/ratio indicated it is possible to give the

form

$$\bar{q} = \frac{1+\xi^2}{12(1-\nu^2)}\eta + \frac{\xi^4}{(1+\xi^2)\eta}.$$

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The minimum value $\bar{q}$ answers upper critical load. If the parameter

$$\varepsilon = \frac{R\delta}{L^2}$$

is low, then

$$\min \bar{q} = \bar{q}_e = \frac{4\pi\sqrt{\varepsilon}}{(36(1-\nu^2))^{3/4}} \simeq 0{,}92\sqrt{\varepsilon}.$$

This minimum value $\bar{q}$ is obtained for m=1 and n, determined by the relationship/ratio

$$\left(\frac{\pi R}{Ln}\right)^8 = \frac{\pi^4\varepsilon^2}{36(1-\nu^2)}. \qquad (**)$$

Since the character of the periodicity of sagging/deflections in supercritical deformation is retained, the value n, which enters into the expression of strain energy, it is determined by relationship/ratio (**). Substituting the value of n in formula for U, we will obtain

$$U = \frac{\pi^3 E\delta^3}{24(1-\nu^2)n^2}\left(\frac{R}{L}\right)\left\{\frac{\pi\sqrt{\varepsilon}}{\sqrt{6}}\int_{-1}^{1} y''^2 dx + 2\nu\int_{-1}^{1} y'^2 dx + \right.$$
$$\left. + 6(1-\nu^2)\frac{c}{\sqrt{\varepsilon}}\int_{-1}^{1} |y''|^{1/2} dx\right\} + \text{const.}$$

Let us find that produced by the external pressure q work by the deformation of shell. A change in the volume, bounded by shell, is equal

$$\Delta V = \frac{\pi^3 R^2 L}{8n^2} \int_{-1}^{1} y^2\,dx + \text{const.}$$

Work

$$A = q\,\Delta V.$$

Hence

$$A = \frac{\pi^3 R^2 L q}{8n^2} \int_{-1}^{1} y^2\,dx + \text{const.}$$

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Or, by introducing dimensionless quantity $\bar{q}$, connected with external pressure by the **relationship/ratio**

$$q = \bar{q} E \frac{\delta^2}{R^2},$$

we will obtain

$$A = \frac{\pi^3 \bar{q} E \delta^3}{8n^2} \frac{1}{\varepsilon} \left(\frac{R}{L}\right) \int_{-1}^{1} y^2\,dx + \text{const.}$$

2. Investigation of supercritical ones by deformation of **unlimitedly elastic shells under external pressure.** Let us assume

$$J = \frac{\pi \sqrt{\varepsilon}}{\sqrt{6}} \int_{-1}^{1} y''^2\,dx + 2\nu \int_{-1}^{1} y'^2\,dx + 6(1-\nu^2)\frac{c}{\sqrt{\varepsilon}} \int_{-1}^{1} |y''|^{1/2}\,dx,$$

$$\lambda = \int_{-1}^{1} y^2\,dx.$$

Then

$$U = \frac{\pi^3 E \delta^3}{24(1-\nu^2)n^2} \left(\frac{R}{L}\right) J + \text{const},$$

$$A = \frac{\pi^3 \bar{q} E \delta^3}{8n^2 \varepsilon} \left(\frac{R}{L}\right) \lambda + \text{const.}$$

The true form of shell in the state of equilibrium under load q we will determine from the condition of the minimum of functional J under the condition of stability λ. For solving this isoperimetric task, let us use an approximate method of solution, as in the case of axial compression in §1.

Function $y(x)$, that realizes the minimum of functional J, we will search for in the class of the smooth functions, determined by the conditions

$$y'' = \begin{cases} 0 & \text{при}^{(1)} |x| \geqslant \alpha, \\ a & \text{при}^{(1)} |x| < \alpha. \end{cases}$$

Key:(1). with.

where a and α - varied parameters. The graph of this function is represented in Fig. 43. It consists of parabola in section $(-\alpha, \alpha)$ and two smoothly adjacent it rectilinear cuts. This selection of function $y(x)$ is predicted by the obvious symmetry of function, realizing minimum J, and by its inversion into zero together with the second derivatives at the ends of interval (-1, 1).

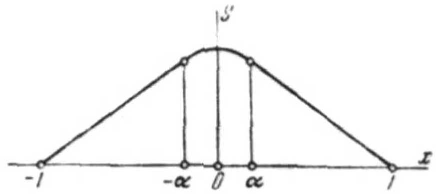

Fig. 43.

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Analytical expression of function $y(x)$ following:

$$y(x)=\begin{cases} \frac{ax^2}{2}-\frac{a\alpha^2}{2}+a\alpha(1-\alpha), & |x|\leqslant\alpha, \\ a\alpha(1-x), & |x|>\alpha. \end{cases}$$

It is possible to show which for the function, which realizes the minimum of functional J, the value of the parameter α is small together with ε. Therefore

$$\int_{-1}^{1} y''^2\,dx = 2a\alpha^2,$$

$$\int_{-1}^{1} y'^2\,dx = 2\alpha^2a^2(1+*),$$

$$\int_{-1}^{1} |y''|^{1/2}\,dx = 2a^{1/2}\alpha,$$

$$\int_{-1}^{1} y^2\,dx = \frac{2}{3}\alpha^2a^2(1+*),$$

where the chain wheel designated the expressions, which have the

subordinate value with small α.

Substituting the value of the obtained integrals, into expressions J and λ and disregarding terms *, let us have

$$J=\frac{\pi\sqrt{\varepsilon}}{\sqrt{6}}\,2\alpha a^2+4\nu\alpha^2a^2+12(1-\nu^2)\frac{c}{\sqrt{\varepsilon}}\,\alpha\sqrt{a},$$

$$\lambda=\frac{2}{3}\,a^2\alpha^2.$$

Introducing into expression J value λ instead of α, we will obtain J as the function only of one parameter a

$$J=(\pi\sqrt{\lambda\varepsilon})a+6\nu\lambda+\left(12(1-\nu^2)c\sqrt{\frac{3}{2}}\sqrt{\frac{\lambda}{\varepsilon}}\right)\frac{1}{\sqrt{a}}.$$

When $\nu=0{,}3$ and c=0.19 will be

$$J=(\pi\sqrt{\lambda\varepsilon})\,a+1{,}8\lambda+2{,}5\sqrt{\frac{\lambda}{\varepsilon}}\,\frac{1}{\sqrt{a}}.$$

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It is easy to see that during variation from parameter a

$$\min J=5{,}1\varepsilon^{-1/6}\sqrt{\lambda}-1{,}8\lambda,$$

and this minimum is reached at

$$a=0{,}54\varepsilon^{-2/3}.$$

For a shell, which is found in the state of elastic equilibrium,

$$\frac{dU}{d\lambda}=\frac{dA}{d\lambda}.$$

Taking into account the obtained expression for $\min J$, from which we find the value of the external pressure $\bar{q}$ in dependence on the common/general/total deformation of shell, which we characterize by the parameter λ:

$$\bar{q}=\varepsilon\left(0{,}95\,\frac{\varepsilon^{-1/_8}}{\sqrt{\lambda}}+0{,}67\right),\quad \varepsilon=\frac{R\delta}{L^2}.$$

We see that the received by shell load is decreased during an increase in deformation (λ). The states of equilibrium of shell are unstable. This will be in complete agreement with the data of the experimental study of shells under external pressure: the transition to supercritical deformation of shell as a result of loss of stability it occurs without an increase in the external load.

Let us determine maximum geometrically permissible deformation (λ). Since $|y(x)|$ according to sense is not more than unity, and $\max y(x)$ is reached at x=0, then

$$\left|-\frac{a\alpha^2}{2}+a\alpha(1-\alpha)\right|\leqslant 1.$$

Value

$$\lambda\simeq\frac{2}{3}\,a^2\alpha^2.$$

Hence, taking into account the inequality indicated, with small ε (consequently, small α), we obtain

$$\max\lambda\simeq\frac{2}{3}.$$

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Substituting this value λ in formula for $\bar{q}$, we will obtain the value of the lower critical load

$$\bar{q}_l = \varepsilon(1{,}16\varepsilon^{-1/6} + 0{,}67), \quad \varepsilon = \frac{R\delta}{L^2}.$$

Thus, lower critical load for the hinged supported on edges cylindrical shell under external pressure is determined from the formula

$$q_l = \bar{q}_l E \frac{\delta^2}{R^2},$$

where

$$q_l = \varepsilon(1{,}16\varepsilon^{-1/6} + 0{,}67), \quad \varepsilon = \frac{R\delta}{L^2},$$

R - a radius of shell, δ - thickness, E - modulus of elasticity.

Comparing the obtained value $\bar{q}_i$ with the value of the upper critical load

$$\bar{q}_e = 0{,}92\sqrt{\varepsilon},$$

let us have

$$\bar{q}_i = \bar{q}_e(1{,}26\varepsilon^{1/3} + 0{,}73\varepsilon^{1/2}).$$

Hence it is apparent that at the low value of the parameter ε (but we assume similar little) the lower critical load of unlimitedly

elastic shell can be considerably less than the upper critical load.

3. Effect of initial bending of shell on stability at external pressure. Shells with the limited elasticity of material. As in the case of axial compression, the imperfection of the geometric form of shell can influence the value of critical pressure, with which the shell loses stability. Now we will examine the influence of the initial bending of shell on upper critical load at an external pressure. In this case, we will proceed from assumption about the fact that the effect of initial bending will be greatest, when it reproduces the form of shell during supercritical deformation.

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Let us establish the relation between the parameters λ, which characterizes supercritical deformation, and value Δ of the maximum transverse sagging/deflection of shell at this deformation. In initial the variables X, y we have

$$\Delta = \frac{\pi}{2n} y_{\max}.$$

After the standardization of variables on the formulas

$$y = \frac{a}{2}\bar{y}, \quad x = \frac{L}{2}\bar{x},$$

we will obtain

$$\Delta = \frac{\pi}{2n}\frac{a}{2}\bar{y}_{\max}.$$

Let parameter

$$\lambda = \int_{-1}^{1} \bar{y}^2 \, d\bar{x}.$$

In view of the specific form of plotted function $\bar{y}(x)$ (Fig. 43), it is possible to count that

$$\lambda \simeq \frac{2}{3}\bar{y}^2_{max}.$$

If we here introduce instead of $\bar{y}_{max}$ sagging/deflection Δ, then we will obtain

$$\lambda = \frac{32n^2\Delta^2}{3\pi^2 a^2} = \frac{32n^4}{3\pi^4 R^2}\Delta^2$$

or, since n is determined by the condition

$$\left(\frac{\pi R}{Ln}\right)^8 = \frac{\pi^4\varepsilon^2}{36(1-\nu^2)},$$

then

$$\lambda = 6,1\frac{\Delta^2 R}{L^2\delta}.$$

Substituting this value λ in formula for the received by shell load during assigned/prescribed deformation (λ), we will obtain the evaluation of the effect of the initial bending of value Δ on loss of stability at the external pressure

$$\bar{q} = \varepsilon\left(0,34\varepsilon^{-1/4}\frac{L}{\Delta}\sqrt{\frac{\delta}{R}} + 0,67\right).$$

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This formula, naturally, cannot be used at relatively low values Δ, since her conclusion is based on the examination of deformations with a considerable change in the form of shell.

Our examinations in p. 2 were related to unlimitedly elastic

shells. For shells with the limited elasticity of material, they are used only when there the deformations in question do not lead to the voltage/stresses, which emerge beyond elastic limit of the material of shell. Let us explain, are such these shells, that is what conditions they must satisfy their geometric dimensions and mechanical characteristics.

Maximum voltage/stresses σ on the surface of shell during supercritical deformation appear in the zone of powerful local bending and are determined from the formula

$$\sigma = c'E\left(\frac{1}{\rho}\right)^{1/2}\delta^{1/2}\alpha^{3/2}.$$

For the deformations

$$\alpha = \frac{\pi}{2n}, \quad \frac{1}{\rho} = \frac{2\pi R}{nL^2}\,|y''|,$$

$$\max\frac{1}{\rho} = \frac{2\pi R}{nL^2}\,a.$$

In p. 2, it is shown, that parameter a in entire time of supercritical deformation retains one and the same constant value

$$a = 0{,}54\varepsilon^{-2/3}, \quad \varepsilon = \frac{R\delta}{L^2}.$$

Therefore maximum voltage/stresses from bending on the surface of shell are constant and equal to

$$\sigma = c'E\left(\frac{2\pi R}{nL^2}\,0{,}54\varepsilon^{-2/3}\right)^{1/2}\delta^{1/2}\left(\frac{\pi}{2n}\right)^{3/2}.$$

Substituting here value of n, determined by the relationship/ratio

$$\left(\frac{\pi R}{nL}\right)^8 = \frac{\pi^4}{36\,(1-\nu^2)}\left(\frac{R\delta}{L^2}\right)^2,$$

we will obtain

$$\sigma = 0{,}4E\,\frac{\delta}{R}\left(\frac{\delta R}{L^2}\right)^{-1/6}.$$

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Thus, the examinations p. 2 can be used to shells with the limited elasticity, in particular, formula for the lower critical load

$$\bar{q}_i = \varepsilon(1{,}16\varepsilon^{-1/_4} + 0{,}67)$$

it is possible to use, if voltage/stresses by value $0{,}4E\frac{\delta}{R}\left(\frac{\delta R}{L^2}\right)^{-1/_4}$ do not cause plastic deformations in the material of shell. We will consider this condition as that carried out, if

$$0{,}4E\frac{\delta}{R}\left(\frac{\delta R}{L^2}\right)^{-1/_4} < \sigma_в,$$

where $\sigma_в$ - time/temporary strength of materials of shell.

In order to visualize the class of shells which include the obtained result, let us examine an example. We have

$$E = 2 \cdot 10^6 \text{ кг/см}^2, \quad \sigma_в = 4 \cdot 10^3 \text{ кг/см}^2.$$

Key: (1). kg/cm².

Let us assume that L~R. Then our condition gives

$$\frac{R}{\delta} > 2800.$$

Thus, these are sufficiently films.

Let us examine the now comparatively thick shells, that is the shells for which

$$\sigma_в < 0{,}4E\frac{\delta}{R}\left(\frac{\delta R}{L^2}\right)^{-1/_4}.$$

For such shells, just as in the case of axial compression minimizing functional J, we must observe the supplementary condition: surface stress of shell in the zone of powerful local bending we must not exceed tensile strength $\sigma_{в}$.

In view of the fact that the function

$$J(a)=(\pi\sqrt{\varepsilon\lambda})\,a+1{,}8\lambda+2{,}5\sqrt{\frac{\lambda}{\varepsilon}}\,\frac{1}{\sqrt{a}}$$

has only one minimum, and at the supplementary condition indicated its minimum be reached cannot, then with the observance of this condition will be

$$\min J=J(a_{в}),$$

where $a_{в}$ is determined by the condition of the equality

$$\sigma=\sigma_{в},$$

that is from the relationship/ratio

$$\sigma_{в}=c'E\left(\frac{2\pi R}{L^{2}n}\,a\right)^{1/2}\delta^{1/2}\left(\frac{\pi}{2n}\right)^{3/2}.$$

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Hence

$$\sqrt{a}=\frac{2\sqrt{6}}{\pi c'}\,\frac{\sigma}{E}\,\frac{R}{\delta}=1{,}75\,\frac{\sigma R}{E\delta}.$$

Substituting this value of $\sqrt{a}$ in the expression of functional J, we find the strain energy of the shell

$$U=\frac{\pi^{3}E\delta^{3}}{24(1-\nu^{2})}\left(\frac{R}{L}\right)\left\{9{,}6\sqrt{\varepsilon}\left(\frac{\sigma_{в}R}{E\delta}\right)^{2}\sqrt{\lambda}+\right.$$
$$\left.+1{,}43\left(\frac{E\delta}{\sigma_{в}R}\right)\sqrt{\frac{\lambda}{\varepsilon}}+1{,}8\lambda\right\}.$$

But now, as in the case of unlimitedly elastic shells, from the

condition of the equilibrium

$$\frac{dU}{d\lambda}=\frac{dA}{d\lambda}$$

we find the value of the received by shell load $\bar{q}$ in dependence on the deformation:

$$\bar{q}=\varepsilon\left(1{,}78\sqrt{\frac{\varepsilon}{\lambda}}\left(\frac{\sigma_{e}R}{E\delta}\right)^{2}+0{,}27\left(\frac{E\delta}{\sigma_{e}R}\right)\frac{1}{\sqrt{\lambda\varepsilon}}+0{,}67\right).$$

Lower critical load $\bar{q}_{l}$ corresponds to the maximum allowed value $\lambda\simeq 2/3$. Consequently

$$\bar{q}_{l}=\varepsilon\left(2{,}18\sqrt{\varepsilon}\left(\frac{\sigma_{e}R}{E\delta}\right)^{2}+0{,}33\left(\frac{E\delta}{\sigma_{e}R}\right)\frac{1}{\sqrt{\varepsilon}}+0{,}67\right).$$

In order to make this formula better than foreseeable and compact, let us introduce the parameter

$$\omega=0{,}4\,\frac{E\delta}{\sigma_{e}R}\,\varepsilon^{-1/_{3}}.$$

Then

$$\bar{q}_{l}=\varepsilon\left\{\left(\frac{0{,}35}{\omega^{2}}+0{,}8\omega\right)\varepsilon^{-1/_{6}}+0{,}67\right\}.$$

This formula is related to the shells which have the parameter

$$\omega>1.$$

Let us explain the region of the applicability of the obtained formula.

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For this, let us note that in the beginning of our examination we assumed the parameter α small. For unlimitedly elastic shells this was the consequence of smallness ε. In the case of shells with the limited elasticity in order to use the results of the

preceding/previous examinations, we must assume smallness α. This, of course, will restrict the class of the shells in question.

We have

$$\max \lambda = \frac{2}{3}.$$

Since

$$\lambda = \frac{2}{3} a^2 \alpha^2,$$

that

$$\alpha < \sqrt{\frac{3}{2}} \frac{1}{a}.$$

Substituting here

$$a = \left(1.75 \frac{\sigma_s R}{E\delta}\right)^2,$$

we will obtain

$$\alpha < 0.4 \left(\frac{E\delta}{\sigma_s R}\right)^2.$$

We will consider the condition of smallness α as that carried out, if

$$\frac{E\delta}{\sigma_s R} \leqslant 1.$$

§ 3. Cylindrical shells during twisting.

The investigation of the supercritical elastic states of cylindrical shell during twisting so will be based on the application/use of principle A (chapter 1, § 2). According to this principle the determination of supercritical elastic states is reduced to the solution of variational problem for functional U-A

during the isometric transformations of the initial form of shell. In connection with this we will begin our presentation from the construction of special ones, isometric ones to cylinder, the surfaces with the aid of which let us approach the form of shells during the supercritical deformation, connected with twisting.

1. Special isometric transformation of cylindrical surface. Let us take correct infinite prism with even number n of faces.

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Let α - one of the faces of prism, l - the fin/edge of this face and β - the plane, passing through the axle/axis of prism and the fin/edge l. Let us designate through e the unit vector, directed along the axis of prism, and through e' - vector, perpendicular to plane β. Let us visualize that from point P of face α proceeds ray/beam S, perpendicular to plane β which is reflected from the internal surfaces of faces. It is not difficult to see that this ray/beam after reflection from all faces again will hit point P (Fig. 44). The trajectory of ray/beam is polygon U_P, inscribed into the section/cut of prism, perpendicular to axle/axis. Perimeter u of polygon U_P does not depend on point P.

Let now ray/beam S, which proceeds from point P, have a sense of

the vector $e'+etg\vartheta$. This ray/beam after reflection from all faces of prism will hit certain point P' of face α (Fig. 44). The trajectory of ray/beam PP' is the broken line whose component/links compose with the axle/axis of prism constant angle $\pi/2-\vartheta$. The projection of this broken in the section/cut of prism to a plane, perpendicular to its axis and passing through point P, there is not that another as polygon U_p. Hence it follows that the distance between points P and P' does not depend on point P and is equal $utg\vartheta$, where u - a perimeter of polygon U_p.

Let us take now on the face α of prism infinite into both of sides periodic curve γ with period $utg\vartheta$ let us conduct from each of its points P ray/beam S_p. These ray/beams form certain surface of Z with fin/edges on the faces of prism (Fig. 45). Let us show that this surface is isometric to cylinder.

Really/actually, Z' and Z" - the regular parts of surface Z, which adjoin on the fin/edge γ', which lies at face α'. Surfaces Z' and Z" are cylindrical. Let us continue the forming surfaces Z' for face α'. Then we obtain surface $\overline{Z}'$. This surface is obtained by the mirror reflection of surface Z" relative to plane α'. Hence it follows that surface, comprised of Z' and Z", and the surface, comprised of Z' and $\overline{Z}'$, in the vicinity of fin/edge γ' are isometric.

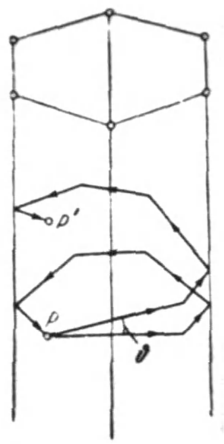

Fig. 44.

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But since the second surface, obviously, is locally isometric plane in the vicinity of fin/edge γ', then this same property possesses the first surface. Thus, surface Z is locally isometric plane everywhere, including on fin/edges. Since surface Z is complete, is locally isometric to plane and topologically equivalent to cylinder, then it is isometric to cylinder. Affirmation is proved. In connection with the forthcoming application/appendices we will now reproduce the construction of surface of Z somewhat a modified form, convenient for use.

Let us take the correct n-angle prism of height/altitude L with the perimeter of basis/base $2\pi R$. let α_1 and α_2 two adjacent faces of

prism and AA' - lateral edge, on which they adjoin. Let us conduct radial plane β through fin/edge AA' and will construct the plane α, parallel to the axle/axis of prism and which separates the faces α_1 and α_2 in half (Fig. 46a).

Let us introduce in plane β the rectangular Cartesian coordinates x, y, after accepting for x axis straight line, on which intersect the planes α and β, but in the origin of coordinates - middle of the segment of this line within prism.

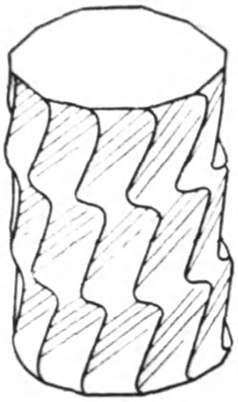

Fig. 45.

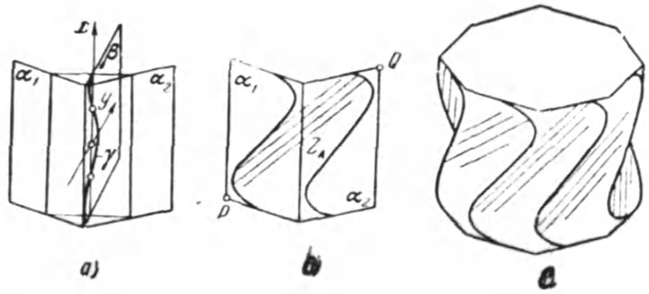

Fig. 46.

Let γ - curve in plane β, given by the equation

$$y = y(x),$$

where y(X) - the function, which satisfies the conditions:

$$1)\ y(x) = y(-x),$$
$$2)\ y\left(x \pm \frac{L}{4}\right) = -y\left(\pm \frac{L}{4} - x\right).$$

Let us conduct through curve γ cylindrical surface with generatrices, parallel to cut PQ, which connects apex/vertexes P and Q of faces α_1 and α_2 (Fig. 46b). The part of this surface,

arrange/located within prism, let us designate Z_A. It is limited by curve γ_1 and γ_2, lying/horizontal at faces α_1 and α_2 respectively. In view of conditions 1) and 2), superimposed for function y(X), the curve γ_1 after rotation about the axle/axis of prism to angle $2\pi/n$ in the appropriate direction is combined with curve γ_2. Hence it follows that if we for each pair of the adjacent faces of prism by the method indicated construct cylindrical surface Z_A, then they form the closed surface of Z (Fig. 46c). Constructed so surface Z is isometric to cylinder.

Let us calculate some values for surface of Z, utilized subsequently. In view of the fact that surface Z is comprised from parts, congruent Z_A, we can be restricted to the examination only of this piece.

First of all, we note that with considerable n for an angle ϑ between forming surfaces Z_A and by the axle/axis of prism we have

$$\vartheta \simeq \frac{4\pi R}{nL}.$$

Let us find normal surface curvature Z_A in the direction, perpendicular by its generatrices. Curvature in radial section is equal to

$$k_r \simeq y''.$$

since $y'^2+1 \simeq 1$. Hence on the Euler formula, normal surface curvature Z_A in the direction, perpendicular to generatrices, will be

$$k \simeq \frac{y''}{\vartheta^2}.$$

Let us determine the curvature of the fin/edges γ_1 and γ_2, which limit region Z_A on surface of Z. For this, let us supplement the system of coordinates x, y in plane β to the three-dimensional system of coordinates x, y, z.

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Then the equation of surface Z_A will be

$$x = u + v\cos\vartheta,$$
$$y = y(u),$$
$$z = v\sin\vartheta.$$

Faces of prism α_1 and α_2 at which lie/rest the curves γ_1 and γ_2, are assigned by the equations

$$\pm\, y = \left(\operatorname{tg}\frac{\pi}{n}\right) z + \text{const.}$$

Curves γ_1 and γ_2 as the intersections of surface Z_A with the faces of prism, are assigned by the matching system of four equations.

Accepting u for the parameter along curves γ_1 and γ_2, we find

$$x' = 1 \pm \frac{y'}{\operatorname{tg}\vartheta \operatorname{tg}\frac{\pi}{n}}, \quad y' = y'(u), \quad z' = \pm \frac{y'}{\operatorname{tg}\frac{\pi}{n}},$$
$$x'' = \pm \frac{y''}{\operatorname{tg}\vartheta \operatorname{tg}\frac{\pi}{n}}, \quad y'' = y''(u), \quad z'' = \pm \frac{y''}{\operatorname{tg}\frac{\pi}{n}}.$$

With small w/n and ϑ for curvature $\varkappa$ curves γ_1 and γ_2 is obtained the formula

$$\varkappa = \frac{n\,|y''|}{\pi\left\{\left(1 \pm \frac{y'n}{\pi\vartheta}\right)^2 + \frac{y'^2 n^2}{\pi^2}\right\}^{3/2}}.$$

Let us find the angle ϕ, formed by the tangential planes of surface Z_A with the planes of curves γ_1 and γ_2, that is by the faces of prism α_1 and α_2. The normals to the planes of the faces of prism have the angular coefficients

$$0, \ \pm 1, \ -\operatorname{tg}\frac{\pi}{n}.$$

The angular coefficients of normal to the surface Z_A are equal to

$$y'\sin\vartheta, \ -\sin\vartheta, \ -y'\cos\vartheta.$$

Hence for an angle ϕ, is obtained the expression

$$\varphi = \frac{\pi}{n}\left\{\left(1 \pm \frac{y'n}{\pi\vartheta}\right)^2 + \frac{y'^2n^2}{\pi^2}\right\}^{1/2}.$$

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The cell/element of the arc of curves γ_1 and γ_2 is equal to

$$ds = \left\{\left(1 \pm \frac{y'n}{\pi\vartheta}\right)^2 + \frac{y'^2n^2}{\pi^2}\right\}^{1/2} du.$$

Let us take points X and X' in curves, which limit surface Z, and arrange/located on one vertical line (Fig. 47). If surface Z isometrically is superimposed on initial cylindrical surface, then points X and X' will not be located on one generatrix. Let us find the angular displacement of point X relative to X' during this imposition.

Let us continue on periodicity surface Z_A for line γ_1. Let X" - the

second end linear generator, that proceeds from point X (Fig. 47). Triangle XX'X" - rectangular with right angle at point X'. If we are run up/turn our cylindrical surface to the plane of triangle XX'X", record/fixing points X and X", then X' will move in the direction of the height/altitude of triangle in certain distance of d. In this case, the distance between points X and X' will increase by value

$$\Delta h = d \sin \vartheta,$$

and the which interests us shift of point X' relatively X will be

$$\delta = 2d \cos \vartheta.$$

For value Δh, we can obtain expression with the aid of function y(X), that assigns curve γ, through which is passed the surface Z_A.

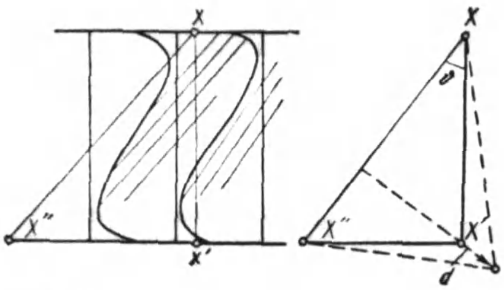

Fig. 47.

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Specifically,, since Δh there is a difference between the distance of points X and X' on surface Z_A and in space, then

$$\Delta h \simeq \frac{1}{2} \int_{-L/2}^{L/2} y'^2 \, dx.$$

Hence for the amount of shift X relative to X' is obtained the following expression:

$$\delta = \frac{1}{\operatorname{tg} \vartheta} \int_{-L/2}^{L/2} y'^2 \, dx,$$

or, since angle ϑ is small, then

$$\delta = \frac{1}{\vartheta} \int_{-L/2}^{L/2} y'^2 \, dx.$$

In the amount of linear shift δ, it is possible to find the angular displacement

$$\omega = \frac{\delta}{R} = \frac{1}{R\vartheta} \int_{-L/2}^{L/2} y'^2 \, dx.$$

Angle ω we will call the angle of twist of cylindrical surface with its bending into surface of Z.

2. Expression for functional W(Z). The determination of the elastic state of shell during supercritical deformation is reduced to the examination of variational problem for the functional

$$W(Z) = U - A,$$

where U - strain energy of shell, A - the produced by external load work. Let us find expression U and A for the isometric transformations of initial cylindrical surface into form of Z. Strain energy

$$U = U_Z + U_\gamma,$$

where U_Z - energy of bending over basic surface, U_γ - strain energy along fin/edges.

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Energy U_Z per the unit surface area of shell is calculated from the formula

$$\bar{U}_Z = \frac{E\delta^3}{24(1-\nu^2)}(\Delta k_1^2 + \Delta k_2^2 + 2\nu \Delta k_1 \Delta k_2).$$

where Δk_1 and Δk_2 - extreme changes in the normal curvatures upon transfer from initial cylindrical surface to surface of Z. In view of

the smallness of angle ϑ, it is possible to count that

$$\Delta k_1 \simeq k - \frac{1}{R}, \quad \Delta k_2 = 0,$$

where k - normal surface curvature Z in direction, perpendicular to its generatrices, and R - radius of initial shell.

Since surface Z is comprised from n of congruent regions Z_A, that

$$U_Z = \iint_Z \bar{U}\, dS = n \iint_{Z_A} \bar{U}\, dS.$$

Designating b(x) the length of the cut of the forming surface Z_A. by passing through the point (x) curve γ, with small ones ϑ and π/n let us have

$$\iint_{Z_A} \bar{U}\, ds \simeq \frac{E\delta^3\vartheta}{24(1-\nu^2)} \int_{-L/2}^{L/2} b(x)\left(k(x) - \frac{1}{R}\right)^2 dx.$$

Length of the cut of generatrix

$$b(x) = \left(\frac{a}{2}\sin\frac{\pi}{n} + y(x)\right)\frac{2}{\sin\vartheta\, \mathrm{tg}\,\frac{\pi}{n}}.$$

Taking into account the smallness of values π/n and ϑ, and also expression for $a=2\pi R/n$, we will obtain

$$b(x) = \left(\frac{\pi^2 R}{n^2} + y(x)\right)\frac{2n}{\pi\vartheta}.$$

Hence

$$\iint_{Z_A} \bar{U}\, ds = \frac{nE\delta^3}{12\pi(1-\nu^2)} \int_{-L/2}^{L/2} \left(\frac{\pi^2 R}{n^2} + y(x)\right)\left(\frac{y''}{\vartheta^2} - \frac{1}{R}\right)^2 dx.$$

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In view of conditions 1) and 2), which satisfies function $y(x)$ (see Section 1), we can write

$$\int_{-L/2}^{L/2} yy''^2\,dx = 0, \quad \int_{-L/2}^{L/2} y''\,dx = 0, \quad \int_{-L/2}^{L/2} y\,dx = 0,$$

$$\int_{-L/2}^{L/2} y''y\,dx = -\int_{-L/2}^{L/2} y'^2\,dx.$$

Therefore

$$\iint_{Z_A} \bar{U}\,dS = \frac{\pi E R\delta^3}{12n(1-\nu^2)\vartheta^4}\int_{-L/2}^{L/2} y''^2\,dx + \frac{nE\delta^3}{6\pi(1-\nu^2)R\vartheta^2}\int_{-L/2}^{L/2} y'^2\,dx + \text{const.}$$

Thus, energy of bending over the basic surface of shell is equal to

$$U_Z = \frac{\pi E R\delta^3}{12(1-\nu^2)\vartheta^4}\int_{-L/2}^{L/2} y''^2\,dx + \frac{n^2E\delta^3}{6\pi(1-\nu^2)R\vartheta^2}\int_{-L/2}^{L/2} y'^2\,dx + \text{const.}$$

Let us calculate now energy U_γ. We have

$$U_\gamma = U_\gamma^0 + \Delta U_\gamma,$$

where

$$U_\gamma^0 = n\int_\gamma cE\delta^{5/2}\varphi^{5/2}\varkappa^{1/2}\,ds_\gamma,$$

$$\Delta U_\gamma = n\frac{E\delta^3}{12(1-\nu^2)}\int_\gamma \varphi(-2k_n + k_e + k_i)\,ds_\gamma.$$

Here φ - angle between the plane curved γ and the tangential planes of surface Z along this curve, k_e and k_i - normal surface curvatures Z in the direction, perpendicular to fin/edge, and k_n -

normal curvature of initial cylindrical surface in the corresponding direction. Let us begin from the determination of value ΔU_γ.

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From the construction of surface of Z, it follows that the normal curvatures k_e and k_i are equal in magnitude and have opposite signs. Therefore $k_i + k_e = 0$. and, therefore,

$$\Delta U_\gamma = n\frac{E\delta^3}{6(1-\nu^2)}\int\limits_\gamma \varphi k_n\, ds_\gamma.$$

If we designate through α the angle between the tangent to curve γ and the direction of the axle/axis of prism, then

$$k_n \simeq \frac{1}{R}\cos^2\alpha.$$

It is not difficult to obtain expression for an angle α, on the basis of equation by curve γ. With small π/n and ϑ we have

$$\cos^2\alpha = \frac{\left(1 \pm \frac{y'n}{\pi\vartheta}\right)^2}{\left(1 \pm \frac{y'n}{\pi\vartheta}\right)^2 + \frac{y'^2 n^2}{\pi^2}}.$$

Angle

$$\varphi = \frac{\pi}{n}\left(\left(1 \pm \frac{y'n}{\pi\vartheta}\right)^2 + \left(\frac{y'n}{\pi}\right)^2\right)^{1/2}.$$

The cell/element of arc to curve γ is equal to

$$ds_\gamma = \left(\left(1 \pm \frac{y'n}{\pi\vartheta}\right)^2 + \left(\frac{y'n}{\pi}\right)^2\right)^{1/2} dx.$$

Substituting these values in formula for ΔU_γ. we will obtain

$$\Delta U_\gamma = -\frac{n^2 E\delta^3}{6\pi(1-\nu^2)R\vartheta^2}\int\limits_{-L/2}^{L/2} y'^2\, dx.$$

Let us calculate now U_γ^0. Substituting the value φ. $\varkappa$ and ds_γ in

formula for U^0_{γ}, let us have for one fin/edge γ_i

$$U^0_{\gamma_i} = \int_{-L/2}^{L/2} cE\delta^{5/2}\left(\frac{\pi}{n}\right)^2 |y''|^{1/2}\left(\left(1 \pm \frac{y'n}{\pi\vartheta}\right)^2 + \left(\frac{y'n}{\pi}\right)^2\right)dx =$$

$$= \int_{-L/2}^{L/2} cE\delta^{5/2}\left(\frac{\pi}{n}\right)^2 |y''|^{1/2}\left(1 + \left(\frac{y'n}{\pi\vartheta}\right)^2 + \left(\frac{y'n}{\pi}\right)^2\right)dx.$$

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In view of smallness ϑ, it will be

$$\left(\frac{y'n}{\pi\vartheta}\right)^2 \gg \left(\frac{y'n}{\pi}\right)^2.$$

Therefore it is possible to count that

$$U^0_{\gamma_i} = \int_{-L/2}^{L/2} cE\delta^{5/2}\left(\frac{\pi}{n}\right)^{1/2} |y''|^{1/2}\left(1 + \left(\frac{y'n}{\pi\vartheta}\right)^2\right)dx.$$

The energy, connected with local bending, on all fin/edges of surface Z is equal to

$$U^0_{\gamma} = cnE\delta^{5/2}\left(\frac{\pi}{n}\right)^2 \int_{-L/2}^{L/2} |y''|^{1/2}\left(1 + \left(\frac{y'n}{\pi\vartheta}\right)^2\right)dx.$$

Store/adding up the obtained formulas on U_Z, U^0_{γ} and ΔU_{γ} we find total energy of the supercritical deformation of the shell

$$U = \frac{\pi ER\delta^3}{12(1-\nu^2)\vartheta^4}\int_{-L/2}^{L/2} y''^2\,dx +$$

$$+ cnE\delta^{5/2}\left(\frac{\pi}{n}\right)^2 \int_{-L/2}^{L/2} |y''|^{1/2}\left(1 + \frac{y'n}{\pi\vartheta}\right)^2\right)dx + \text{const}.$$

Hence, taking into account of condition 1) and 2) for function y(X), we will obtain

$$U = \frac{\pi ER\delta^3}{6(1-\nu^2)\vartheta^4}\int_{-L/4}^{L/4} y''^2\,dx +$$

$$+ 2cnE\delta^{5/2}\left(\frac{\pi}{n}\right)^2 \int_{-L/4}^{L/4} |y''|^{1/2}\left(1 + \left(\frac{y'n}{\pi\vartheta}\right)^2\right)dx + \text{const}.$$

Let us introduce dimensionless variables $\bar{x}$ and $\bar{y}$, set/assuming

$$x=\frac{L\bar{x}}{4}, \quad y=\frac{a}{2}\sin\frac{\pi}{n}\,\bar{y}\simeq\frac{\pi^2 R}{n^2}\,\bar{y}.$$

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In the new variables $\bar{x}$ and $\bar{y}$ the feature above which we will lower, we have

$$U=\frac{\pi E\delta^3}{24\,(1-\nu^2)}\left(\frac{L}{R}\right)\int_{-1}^{1} y''^2\,dx+$$

$$+\,2\pi cE\delta^{5/2}R^{1/2}\left(\frac{\pi}{n}\right)^2\int_{-1}^{1}|\,y''\,|^{1/2}(1+y'^2)\,dx+\text{const}.$$

The angle of twist of shell in new variables is equal to

$$\omega=2\left(\frac{\pi}{n}\right)^3\int_{-1}^{1} y'^2\,dx.$$

Just as in the preceding/previous examinations, we proceed from assumption about the fact that the periodicity of the sagging/deflections of shell during supercritical deformations is determined by the periodicity of sagging/deflections at the moment of loss of stability. Therefore parameter n, entering the expression of strain energy, we will determine from the examination of the loss of stability of shell.

Let the hinged supported on edges cylindrical shell of radius R, of length L and of thickness δ locate under the action of evenly distributed on edges tangential force s. Let us introduce on the surface of shell curvilinear coordinates X, y because this was made in the preceding/previous examinations, and let us designate through w(X, y) the radial sagging/deflection of shell at the moment of loss of stability. Function w(X, y) satisfies the differential equation

$$\frac{D}{\delta}\Delta\Delta\Delta\Delta w+\frac{E}{R^2}\frac{\partial^4 w}{\partial x^4}-2s\Delta\Delta\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)=0.$$

In the linear theory of shells for sagging/deflection w(X, y) usually is accepted the expression of the form

$$w=c\sin\frac{\pi x}{L}\cos\left(\frac{n(y-\gamma x)}{R}\right).$$

If this expression is substituted into equation for w, then we will obtain two relationship/ratios for values n, γ and s.

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The small value s, determined by these relationship/ratios, is equal

$$s_e\simeq 0{,}8E\frac{\delta}{R}\left(\frac{R\delta}{L^2}\right)^{1/4}$$

and it is upper critical load. This value s is obtained with

$$\gamma=\bar{\gamma}\varepsilon^{1/4},\quad n=\frac{\pi R}{\bar{\xi}L}\varepsilon^{-1/4},$$

where

$$\bar{\gamma}\simeq 1{,}8,\quad \bar{\xi}\simeq 0{,}75,\quad \varepsilon=\frac{R\delta}{L^2}.$$

Substituting the obtained value of n in the expression of the strain energy and angle of twist, we will obtain for them the

following resultant expressions:

$$U=\frac{\pi E\delta^3}{24(1-\nu^2)}\left(\frac{L}{R}\right)\int_{-1}^{1} y''^2\,dx+$$
$$+2\pi c\bar{\xi}^2 E\delta^3\left(\frac{L}{R}\right)\int_{-1}^{1}|y''|^{1/2}(1+y'^2)\,dx+\text{const}.$$
$$\omega=2\bar{\xi}^3\left(\frac{L}{R}\right)^3\left(\frac{R\delta}{L^2}\right)^{3/4}\int_{-1}^{1} y'^2\,dx.$$

3. Investigation of supercritical deformations of shell. With $\bar{\xi}=0.75$ we have

$$U=\frac{\pi E\delta^3}{24(1-\nu^2)}\left(\frac{L}{R}\right)J(y),\quad \omega=2\cdot 0{,}75^3\left(\frac{L}{R}\right)^3\left(\frac{R\delta}{L^2}\right)^{3/4}\lambda(y).$$

where

$$J(y)=\int_{-1}^{1}\left(y''^2+4{,}35\,|y''|^{1/2}\left(1+y'^2\right)\right)dx,$$
$$\lambda(y)=\int_{-1}^{1} y'^2\,dx.$$

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Function y(X), the assigning form of shell during supercritical deformation, we will determine from the condition of the minimum of energy of elastic deformation U at the assigned/prescribed angle of twist ω, that is from the condition of minimum J(y) when λ(y)=const. The solution of this variational problem let us search for among functions y(X), that satisfy the conditions:

1) $y(-1)=y(1)=0$;
2) $y''(x)=a$ при (1) $|x|<b<1$;
3) $y''(x)=0$ при (1) $|x|\geqslant b$.

Key: (1). with.

where a and b - varied parameters. Graphically function y(X) is represented in Fig. 48. Graph consists of two rectilinear cuts, which smoothly adjoin the parabola. Analytically function y(X) is assigned by the equations:

$$y(x)=abx-ab \quad \text{при}^{(1)}\ x\geqslant b,$$
$$y(x)=-abx-ab \quad \text{при}^{(1)}\ x\leqslant -b,$$
$$y(x)=\frac{ax^2}{2}-\frac{ab^2}{2}-ab \quad \text{при}^{(1)}\ |x|<b.$$

Key: (1). with.

For the integrals, entering J(y) and λ(y), are obtained the following expressions:

$$\int_{-1}^{1} y''^2\,dx=2a^2b, \quad \int_{-1}^{1} y'^2\,dx=2a^2b^2-\frac{4}{3}a^2b^3,$$
$$\int_{-1}^{1} |y''|^{1/2}(1+y'^2)\,dx=2a^{1/2}b+\frac{2}{3}a^{5/2}b^3.$$

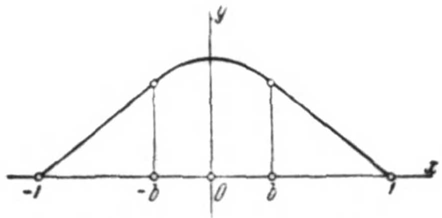

Fig. 48.

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Substituting these values in J and λ, we will obtain

$$J = 2a^2b + 4{,}35\left(2a^{1/2}b + \frac{2}{3}a^{5/2}b^3\right),$$
$$\lambda = 2a^2b^2 - \frac{4}{3}a^2b^3.$$

The task of minimum J under condition λ=const we is solved numerically. Specifically,, record/fixing λ, we are assigned by the different values of parameter b, from condition λ=const, we find the corresponding to them values of a, then we compute J and we determine its small value.

During this solution of task, logically arises the question concerning are such the allowed values λ. In order to solve this question, let us note that the dimensionless variable y according to sense does not exceed unity on module/modulus. But since

$$\max|y| = -\frac{ab^2}{2} + ab,$$

that

$$-\frac{ab^2}{2} + ab \leqslant 1.$$

consequently,

$$\lambda < \left(2 - \frac{4}{3}b\right)\frac{1}{\left(1 - \frac{b}{2}\right)^2} < 2,7.$$

By method indicated above at values λ from interval 1≤λ≤2.8, through every 0.2 were obtained values minJ. In this case, it turned out that the value of the variable b, for which in the interval of values λ indicated was reached minimum J, it does not virtually change, and it it is equal ≈0.3.

Substituting the value b=0.3 in expressions J and λ, we will obtain

$$\lambda = 0,144a^2, \quad J = 0,6a^2 + 4,35a^{1/4}(0,6 + 0,018a^2).$$

Whence

$$J = 4,15\lambda + 7,1\lambda^{1/4}(0,6 + 0,125\lambda).$$

Let us establish communication/connection between the common/general/total deformation of shell which is characterized by the parameter λ, and by the received by shell load.

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Condition of the equilibrium of shell -

$$\frac{dU}{d\lambda} = \frac{dA}{d\lambda},$$

where U(y) - the strain energy, A - the produced by external load

work. Strain energy -

$$U = \frac{\pi E \delta^3}{24(1-\nu^2)} \left(\frac{L}{R}\right) J.$$

Let us find the work A. We have

$$A = 2\pi R\delta s\omega R.$$

Substituting here the values

$$s = \bar{s}E\frac{\delta}{R}, \quad \omega = 2 \cdot 0{,}75^3 \left(\frac{L}{R}\right)^3 \left(\frac{R\delta}{L^2}\right)^{3/4} \lambda,$$

we will obtain

$$A = 1{,}7\pi\bar{s}E\delta^3 \left(\frac{L}{R}\right) \left(\frac{R\delta}{L^2}\right)^{-1/4} \lambda.$$

From the condition of equilibrium, we now find which interests us communication/connection between load and deformation

$$\bar{s} = 0{,}0272(4{,}15 + 1{,}06\lambda^{-3/4} + 1{,}1\lambda^{1/4}) \left(\frac{R\delta}{L^2}\right)^{1/4}.$$

It is easy to see that function $\bar{s}(\lambda)$ monotonically decreases with $\lambda<2.9$, therefore, and despite all allowed values λ. This completely corresponds to the character of transition to supercritical deformations.

Let us find the small value of $\bar{s}$. We have

$$\lambda \leqslant \left(2 - \frac{4}{3}b\right) \frac{1}{\left(1 - \frac{b}{2}\right)^2}.$$

With b=0.3 value $\lambda \leqslant 2.2$. Therefore

$$\min \bar{s} = 0{,}18 \left(\frac{R\delta}{L^2}\right)^{1/4}.$$

Thus, the smallest received by shell load during supercritical deformation, that is lower critical load, is determined from the formula

$$s_l = 0{,}18E\frac{\delta}{R}\left(\frac{R\delta}{L^2}\right)^{1/4}$$

and comprises approximately the fourth of the upper critical load

$$s_e = 0{,}8E\frac{\delta}{R}\left(\frac{R\delta}{L^2}\right)^{1/4}.$$

Our all examinations, until now, were related, strictly speaking, only to unlimitedly elastic shells, since we did not consider that fact that in the zone of powerful local bending were possible the inelastic deformations. Therefore for real shells, i.e., the shells, which possess the limited elasticity, the obtained results are used only with the observance of some conditions. Let us find these conditions.

For maximum voltage/stresses σ in the zone of powerful local bending, we have the formula

$$\sigma = c'E\delta^{1/2}\varphi^{3/2}\varkappa^{1/2}.$$

Substituting here values ϕ and $\varkappa$, we will obtain

$$\sigma = c'E\delta^{1/2}\left(\frac{\pi}{n}\right)|\,y''\,|^{1/2}$$

or, in the dimensionless variables X, y,

$$\sigma = c'E\left(\frac{\pi}{n}\right)^2(16\,|\,y''\,|)^{1/2}\left(\frac{R\delta}{L^2}\right)^{1/2}.$$

Determining the value of parameter n from the relationship/ratio

$$\frac{\pi R}{Ln} = \bar{\xi}\left(\frac{R\delta}{L^2}\right)^{1/4}, \quad \bar{\xi} \simeq 0{,}75,$$

we will obtain

$$\sigma = 2{,}1E\frac{\delta}{R}|\,y''\,|^{1/2}.$$

The great value $|y''|=a$ is obtained at the greatest value $\lambda=2.2$

and is determined from the condition

$$\lambda = 0{,}144a^2.$$

Hence a=3.8, and therefore

$$\sigma \simeq 4E\frac{\delta}{R}.$$

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Thus, our examinations are related only to such shells with the limited elasticity of material, whose voltage/stresses by value 4Eδ/R do not cause plastic deformations. We will consider this condition as that carried out, if

$$4E\frac{\delta}{R} < \sigma_{s}.$$

where σ_{s} - time/temporary strength of materials.

Example. For shell of steel $E=2\cdot10^6$ of kg/cm², $\sigma_{s}=4\cdot10^3$ kg/cm. Our condition is reduced to that, in order to

$$\frac{R}{\delta} > 2000.$$

Thus, the discussion deals with very films.

Now we will examine the supercritical deformations of comparatively thick shells, that is the shells, which satisfy the condition

$$\sigma_{s} < 4E\frac{\delta}{R}.$$

For such shells of the deformations, examine/considered above, they lead to the voltage/stresses in the zone of powerful local bending, which emerge beyond elastic limit of material. In connection with this according to the considerations which already were given in the

preceding/previous paragraphs, minimizing functional J with λ=const, we must place as supplementary requirement the observance of condition $\sigma \leqslant \sigma_{в}$, where α - voltage/stress in the zone of powerful local bending.

We found for bending stresses the expression

$$\sigma = 2{,}1 E \frac{\delta}{R} \sqrt{a}.$$

Hence it follows that at the considerable deformations of comparatively thick shells parameter a retains constant value, precisely,

$$a = \left(\frac{1}{2{,}1} \frac{\sigma_{в} R}{E\delta}\right)^2.$$

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For such deformations the minimum of the functional

$$J = 2a^2 b + 4{,}35 \left(2a^{1/2} b + \frac{2}{3} a^{1/2} b^3\right)$$

is obtained with the substitution of the value

$$a = \left(\frac{1}{2{,}1} \frac{\sigma_{в} R}{E\delta}\right)^2$$

and of value b, which at the value of a indicated is determined by the equality

$$\lambda = 2a^2 b^2 - \frac{4}{3} a^2 b^3.$$

From the condition of the equilibrium

$$\frac{dU}{d\lambda} = \frac{dA}{d\lambda}$$

we find communication/connection between deformation λ and received by load $\bar{s}$. Specifically,.

$$\bar{s} = 0{,}0272 \frac{dJ}{d\lambda}.$$

Fiche #5

Minimizing s from the parameter of deformation, we find its value $\bar{s}_i$, corresponding to lower critical load. Figure 49 graphically depicts the dependence of $\bar{s}_i$ on the elasto-plastic properties of material.

The presentation of this paragraph can be summed up by following conclusion.

The received by shell load during twisting in the course of supercritical deformation is decreased. If the geometric parameters of shell and the mechanical characteristics of material satisfy the condition

$$4E\frac{\delta}{R}\leqslant\sigma_{s}.$$

then the smallest received by shell load, that is lower critical load, is determined from the formula

$$s_i=0{,}18E\frac{\delta}{R}\left(\frac{R\delta}{L^2}\right)^{1/4}.$$

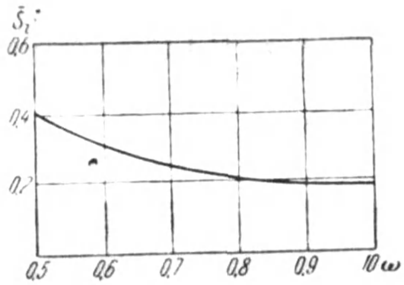

Fig. 49.

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But if the condition indicated is not satisfied, that is if

$$4E\frac{\delta}{R} > \sigma_e,$$

then

$$s_i = \bar{s}_i E \frac{\delta}{R}\left(\frac{R\delta}{L^2}\right)^{1/4},$$

where the coefficient $\bar{s}_i$ depends on the parameter

$$\omega = \frac{1}{4}\frac{\sigma_e R}{E\delta}.$$

This dependence is represented in Fig. 49.

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SUPPLEMENT I
SOME QUESTIONS OF DYNAMICS.

The methods of the study of the supercritical elastic states of shells with static loading, developed in chapter 1, 2, 3, can be used also during the solution of the tasks of dynamics. In present supplement the application/use of these methods is illustrated on the specific problems of the stability of shells with dynamic load, and also during the study of oscillations with large amplitude.

§ 1. Strictly convex shells with dynamic load.

We will examine the flat strictly convex hull, rigidly attached on edge. This shell under the external pressure q_0, greater than critical (p_e). loses stability and it begins to be swelled. The process of bulge rapidly progresses and it leads to the complete cracking of shell which is accompanied by "cotton/knock". To the study of the dynamics of this process it will be dedicated to p. 1.

If to shell, which is located under the external pressure q_0 smaller than critical, affects intermittent load q, much greater critical, then the loss of stability of shell, caused by load q,

because of load q_0 can also lead to cracking of shell. The determination of the minimum momentum/impulse/pulse q_T, capable of causing cracking shell, which is located under pressure q_0, will be given in p. 2.

1. Dynamics of "cotton/knock" with uniform loading of shell.

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Let us visualize sufficiently flat strictly convex shell, rigidly attached on edge, on which affects the evenly distributed external pressure $p > p_e$. Under this pressure the shell is swelled and, thus, it starts up. The deformation of shell is completed by the complete cracking of shell with characteristic "cotton/knock". The task, which we want to examine, consists in the study of the deformation of shell on time, in particular in the explanation of the physical cause for "cotton/knock".

Qualitatively the deformation of shell we will visualize in the manner that this is represented in Fig. 50a, where are given normal sections of shell at successive moments of time. Figure 50b the same process depicts schematically with respect to the accepted by us method of approaching the form of shell. The deformation of shell up to torque/moment t we visualize in the form of twofold mirror bulge

(Fig. 50c). It consists of the mirror reflection of the segment of shell relative to plane α, close to edge, with the subsequent reflection relative to plane β.

In order to determine the motion of shell in the deformation in question, we will use Hamilton-Ostrogradskiy's variation principle, according to whom a variation in the functional

$$J=\int W\,dt,\quad W=K-U+A,$$

is equal to zero. Here K - the kinetic kinetic energy, U - the strain energy of shell, A - work, produced by the external pressure p. Since us interests a question concerning the physical cause for "cotton/knock", we will be restricted to the study of motion in the final stage where is observed the phenomenon of "cotton/knock".

Let us determine values add/composed by K, U and A in integrand W. Let us begin from strain energy U.

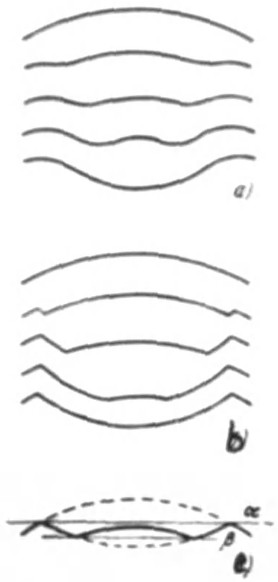

Fig. 50.

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If the planes α and β, which are determining mirror bulge, are sufficiently distant from each other (but us interests now precisely this case), then

$$U = \pi c E \delta^{5/2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)(2h)^{3/2} + (*),$$

where h - height/altitude of the segment, intercept/detached by plane β, R_1 and R_2 - main radii of curvature in the center of bulge, δ - thickness of shell. By sign (*) markedly analogous add/composed, which corresponds to plane α. Under our assumption about the character of the deformation in question this expression is stationary, and therefore it is unessential.

Work A, produced by the external pressure p by the deformation of shell, with an accuracy to unessential term/component/addend, which we also will designate (*), it is determined from the formula

$$A = -2pV + (*).$$

where V - a volume of the segment, intercept/detached by plane β, i.e.

$$V = \pi h^2 \sqrt{R_1 R_2}.$$

The kinetic kinetic energy of shell is equal to

$$K = \frac{1}{2} S \delta \gamma (2h')^2.$$

where S - an area of the segment, intercept/detached by plane β, δ - thickness of shell, γ - material density, 2h' - deformation rate. We have

$$S \simeq 2\pi h \sqrt{R_1 R_2}.$$

Substituting the obtained values into expression W, we will obtain

$$W = 4\pi h \sqrt{R_1 R_2}\, \delta \gamma h'^2 - \pi c E \delta^{5/2} \left(\frac{1}{R_1} + \frac{1}{R_2}\right)(2h)^{3/2} - \\ - 2\pi p h^2 \sqrt{R_1 R_2} + (*).$$

Thus, expression W takes the form

$$W = c_0 \left(h h'^2 - c_1 h^{3/2} - c_2 h^2\right) + (*).$$

where c_0, c_1, c_2 - positive constants (they do not depend on h).

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From inversion into zero variations in functional J, it follows that function h(t), that assigns deformation, satisfies the equation of Eylera - Lagrange

$$2hh'' + h'^2 + \frac{3}{2}c_1\sqrt{h} + 2c_2 h = 0.$$

Multiplying this equation on h' and integrating, we will obtain

$$hh'^2 + c_1 h^{3/2} + c_2 h^2 = c = \text{const}.$$

Integration constant c, obviously, is more than zero.

At the moment of complete cracking ($h \to 0$) $h' \to \infty$. But since h according to sense is non-negative, then the speed of bulge (h') at the moment of cracking suffers breakage. Physically this means that the cracking is accompanied by shock. As a result of this shock is obtained the "cotton/knock".

2. Critical momentum/impulse/pulse. Let us examine flat strictly convex hull, rigidly attached on edge, which is located under the external pressure q_0 smaller than the upper critical value q_r. Let this shell experience/test the short-term external pressure q, which considerably exceeds q_r. Then at sufficient intensity of momentum/impulse/pulse qr, created by this load, the most complete possible cracking of shell. Let us estimate the value of this momentum/impulse/pulse.

We will visualize the deformation of shell under the power influence indicated on it as follows (Fig. 51). At the initial moment under load q, the shell loses stability and it begins to be swelled (1). Then bulge increases and it leads to partial cracking (3). After this the shell either reduces its initial form (bulge disappears), or at some torque/moment (4) deformation stops, and then again increases before complete cracking. Us interests that case when is realized the second of the possibilities indicated.

Fig. 51.

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In view of the short duration of the action of load q, it communicates the region of bulge speed v, which is determined by the relationship/ratio

$$S\delta\gamma v = Sq\tau,$$

where S - an area of the region of bulge, δ - thickness of shell, γ - material density. Hence

$$v = \frac{q\tau}{\gamma\delta}.$$

The kinetic kinetic energy of shell, caused by the effect of momentum/impulse/pulse qτ, is equal to

$$K = \frac{S(q\tau)^2}{2\gamma\delta}.$$

This energy, just as the produced by pressure q_0 work A, in state (4) they transfer/convert into the strain energy of shell. Thus, we have the equality

$$U(4) = K + A(4). \qquad (*)$$

where (4) it indicates that the corresponding value is determined for

the state of shell (4), when the deformation rate is equal to zero.

In view of the instability of the elastic state of equilibrium in the form (4) (chapter 1), the shell from this state either reduces its initial form or it transfer/converts to complete cracking. Everything depends on that, there will be load q(4), received by shell able (4), is less than q_0 or more q_0.

Depending on sagging/deflection 2h in the center of bulge, the strain energy is determined from the formula

$$U=\pi c E \delta^{5/2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)(2h)^{1/2}.$$

The produced by the external pressure q_0 work is equal to

$$A=2\pi q_0 h^2 \sqrt{R_1 R_2}.$$

Substituting these values in relationship/ratio (*), we will obtain equation for sagging/deflection 2h in state (4).

The received by shell load with bulge depending on sagging in center (2h) is limited on the formula

$$q(2h)=\frac{3}{2}cE\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{1}{\sqrt{R_1 R_2}}\frac{\delta^{5/2}}{\sqrt{2h}}.$$

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The minimum value q_0, at which is possible cracking shell of state (4), it is determined by the condition

$$q_0 = q(2h). \qquad (**)$$

where 2h it is found from equation (*). Therefore the minimum momentum/impulse/pulse $q\tau$ depending on static load q_0 we will obtain, if we exclude parameter h from equations (*), (**) and the obtained relationship/ratio is solved relative to $q\tau$.

Let us designate through q_l lower critical pressure for a shell. In chapter 1 (page 55) for it is obtained the formula

$$q_l = \frac{3}{2} cE\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{1}{\sqrt{R_1 R_2}}\frac{\delta^{5/2}}{\sqrt{2h_l}}.$$

where $2h_l$ - the maximum geometrically permissible sagging/deflection in the center of bulge. Values U and A in relationship/ratio (*) can be written thus:

$$U=\frac{2}{3}\pi q_i\sqrt{R_1R_2}(2h)^{3/2}\sqrt{2h_i}.$$

$$A=\frac{\pi}{2}q_i\sqrt{R_1R_2}(2h)^{3/2}\sqrt{2h_i}.$$

In expression for A, we used equality $q_0=q(2h)$.

Now relationship/ratios (*) and (**), from which it is to exclude h, they will be written as follows:

$$\frac{\pi}{6}q_i\sqrt{R_1R_2}(2h)^{3/2}\sqrt{2h_i}=K,$$

$$q_i\sqrt{\frac{h_i}{h}}=q_0.$$

Parameter h easily is eliminated, and we obtain

$$\frac{\pi}{6}q_i^4\sqrt{R_1R_2}(2h_i)^2=Kq_0^3.$$

After substituting into this relationship/ratio the value

$$K=\frac{S(q\tau)^2}{2\gamma\delta},$$

let us have

$$\frac{\pi}{6}q_i^4\sqrt{R_1R_2}(2h_i)^2=\frac{S(q\tau)^2}{2\gamma\delta}q_0^3.$$

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Hence it is apparent that momentum/impulse/pulse $q\tau$ will be minimum,

if is maximal S, that is the area of the region of bulge at the moment of loss of stability. Being segment, this region will be greatest at height/altitude $2h_i$. In this case

$$S = 2\pi h_i \sqrt{R_1 R_2}.$$

Substituting this value in the obtained above relationship/ratio, we find

$$\frac{1}{3} q_i^4 (2h_i) = \frac{(q\tau)^2 q_0^3}{2\gamma\delta}.$$

This formula establishes communication/connection between the value of the static external pressure q_0 on shell and intermittent load q, which effects for time τ, capable of causing the complete cracking of shell. Let us recall that here q_i - lower critical pressure, as we it determined in chapter 1, $2h_i$ - the maximum geometrically permissible sagging/deflection of shell with mirror bulge, δ - thickness of shell, γ - material density.

§ 2. The dynamic load of cylindrical shell.

In present paragraph we will examine the tasks, analogous to that which in § 1 are solved for strictly convex hulls. Let us visualize cylindrical shell, hinged rested on edges, which is located under the action of the axial compressive load q_0 smaller than the upper critical value. Let on this shell short-term affect load q,

←——————————which considerably exceeds upper critical. In this case, the shell loses stability and it begins to be swelled. At sufficient intensity of the momentum/impulse/pulse of load q, it can happen, that the shell does not reduce initial cylindrical form. The determination of this minimum momentum/impulse/pulse composes the content p. 1. In p. 2 is examined analogous task for the momentum/impulse/pulse, created by external pressure.

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1. Critical momentum/impulse/pulse during axial compression. Let us examine the cylindrical shell of radius R, of length L and thickness δ, hinged supported on edges, which is located under the action of the axial compressive load q_0. Furthermore, on shell short-term affects load q for a period of time τ. Let us explain, at what value q_0 the shell, after losing stability under the action of load q, reduces its initial cylindrical form. In connection with this let us examine, first of all, the initial stage of the deformation of shell in time.

We will assume that the bulge of shell at the initial moment is described by the law

$$w = u(t)\sin\frac{2\pi mx}{L}\sin\frac{ny}{R}.$$

Here w - the normal sagging/deflection of shell, m and n - the integral parameters, which characterize the form of wave formation on the surface of shell with bulge, x, y - coordinate on the surface of the shell: x - on generatrix, y - according to the circular section/cut, perpendicular to axle/axis. On the basis of the assumption indicated about the character of the sagging/deflections of shell with bulge, let us find function u(t), that assigns a change in the sagging/deflection with time. For this purpose, let us compose the equation of motion of shell.

The motion of shell is determined from stability condition of the functional

$$J = \int (K - U + A)\,dt,$$

where K - a kinetic energy, of motion, U - strain energy, A - the produced by external load work. Let us find expressions for K, U and A. We will assume that kinetic energy of the motion of shell is caused in essence by radial displacements. Then it is calculated from the formula

$$K = \frac{1}{2}\gamma\delta \int\int w'^2\,dx\,dy,$$

where γ - material density of shell. Simple calculation shows that

$$K=\frac{1}{4}\pi RL\gamma\delta u'^2.$$

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In order to determine strain energy U, let us discuss as follows. First of all, it is obvious,

$$U=Cu^2,$$

where C does not depend on u. Let us compose the equation of the natural oscillations of shell. This there will be the equation of Euler - Lagrange for the functional

$$\int (K-U)\,dt$$

and, therefore, it takes the form

$$\frac{1}{4}\pi RL\gamma\delta u''+Cu=0.$$

Natural vibration frequency $\omega(m, n)$ is expressed as the coefficients of this equation. Specifically,

$$\omega^2=\frac{4C}{\pi RL\gamma\delta}.$$

Hence

$$C=\frac{1}{4}\omega^2\pi RL\gamma\delta.$$

Thus, the strain energy of shell is equal to

$$U=\frac{1}{4}\omega^2\pi RL\gamma\delta u^2.$$

where ω - the natural vibration frequency of shell, which corresponds to the form of wave formation with parameters m and n.

Let us find that now produced by the external load q work A. The lateral deformation of shell is accompanied by axial compression along generatrices. In section/cut y, this compression is equal

$$\Delta L=\frac{1}{2}\int_0^L w_x^2\,dx.$$

Hence the work

$$A=\int_0^{2\pi R}\Delta Lq\delta\,dy=\frac{q\delta}{2}\int\int w_x^2\,dx\,dy.$$

Simple computations give

$$A=\frac{q\pi^3\delta m^2Ru^2}{L}.$$

Now it is easy to write the equation of motion of shell.

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It takes the form

$$u'' - \bar{\omega}^2 u = 0,$$

where

$$-\bar{\omega}^2 = \omega^2 - \frac{4\pi^2 q m^2}{L^2 \gamma}.$$

We assume load q sufficient large, so that

$$\omega^2 - \frac{4\pi^2 q m^2}{L^2 \gamma} < 0.$$

Since sagging/deflection u at the initial moment is equal to zero, the which interests us solution of equation of motion takes the form

$$u = c\,\mathrm{sh}\,\bar{\omega}t.$$

So that would occur the bulge, c must be excellently from zero. In this case, $u'(0) = c\bar{\omega} \neq 0$. Thus, the bulge of shell at the initial moment has different from zero speeds. This, by the way, is explained the "cotton/knock", by which is accompanied the loss of stability.

In order to determine the constant c and, therefore, the deformation rate at the moment of loss of stability, let us discuss as follows. In view of the fact that load q is considerably more than critical, the strain energy of compression force q at the moment of bulge is free/released and transfer/converts into kinetic

energy of motion. From this condition it is possible to determine kinetic energy, therefore, and the deformation rate.

The compression of shell, caused by load q, is equal

$$\Delta L = \frac{q}{E} L.$$

The total effort/force, which effects on the edge of shell and created by load q, will be

$$F = 2\pi R \delta q.$$

Hence energy of shell, communicated by load q, is equal to

$$U_0 = \frac{q^2}{E} 2\pi R L \delta.$$

Since as a result of loss of stability energy U_0 transfer/converts into kinetic energy of bulge, then

$$\frac{q^2}{E} 2\pi R L \gamma = \frac{1}{4} \pi R L \gamma \delta u'^2.$$

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Hence

$$u' = \frac{q \cdot 2\sqrt{2}}{\sqrt{\gamma E}}.$$

Consequently, it is constant

$$c = \frac{q \cdot 2\sqrt{2}}{\bar{\omega}\sqrt{\gamma E}}.$$

Thus, in initial stage bulge occurs according to the law

$$u = \frac{q\,\mathrm{sh}\,\bar{\omega}t}{\bar{\omega}\sqrt{\gamma E}} \cdot 2\sqrt{2}.$$

Up to the torque/moment τ of the break-down of load q, the total energy of shell (kinetic energy and strain energy) will be

$$V = \frac{1}{4}\pi RL\gamma\,\delta u'^2 + \frac{1}{4}\omega^2\pi RL\gamma\,\delta u^2 = \\ = 2\pi RL\delta\,\frac{q^2}{E}\left(\mathrm{ch}^2\,\bar{\omega}\tau + \frac{\omega^2}{\bar{\omega}^2}\,\mathrm{sh}^2\,\bar{\omega}\tau\right).$$

After the break-down of load q, the shell continues to be swelled under load q_0. Let us study this bulge in time. When the deformation of shell becomes considerable, its form it is possible to approach by surface Z, which is described in detail in chapter 3. This surface is determined by function $\tilde{y}(x, t)$. The speed of points during the deformation of surface Z will be $\tilde{y}_t = \frac{\partial \tilde{y}}{\partial t}$. Therefore kinetic energy of motion is equal to

$$K = 2\pi R\,\delta\gamma \int_0^L \tilde{y}_t^2\,dx.$$

Just as in chapter 3, instead of function $\tilde{y}$ let us introduce function $y(x, t)$, set/assuming

$$\tilde{y} = \frac{\pi}{2n}\,y.$$

In this case,

$$K = \frac{\pi^3}{2n^2} R\,\delta\gamma \int_0^L y_t^2\,dx.$$

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With the aid of function $y(x, t)$ the strain energy of shell is expressed on formula (chapter 3)

$$U = \frac{D\pi^2 a}{4n} \int_0^L y_{xx}^2\,dx + \frac{2\nu D\pi}{R} \int_0^L y_x^2\,dx +$$

$$+ 2cnE\delta^{5/2}\left(\frac{\pi}{2n}\right)^{5/2} \int_0^L |y_{xx}|^{1/2}(1 + y_x^2)\,dx + \frac{D\pi L}{R}.$$

The produced by load q_0 work is equal to

$$A = 2\pi R\,\delta q_0 \frac{\pi^2}{8n^2} \int_0^L y_x^2\,dx.$$

Let a and b - be periods of wave formation on the surface of shell in circumference and on generatrix respectively. If ratio b/a is small, then for function $y(x, t)$, that assigns the shape of surface of shell with large sagging/deflections it is possible to accept the expression

$$y = u(t) \sin\frac{2\pi m x}{L}.$$

In this case, for the kinetic energy K, the strain energy U and of

work A are obtained the expressions of the form

$$K = A_1 u'^2,$$
$$U = B_1 u^2 + B_2 |u|^{5/2} + B_3 |u|^{4/3} + B_0,$$
$$A = C u^2 q_0,$$

where A_1, B_1, ... C - the constants which easily are calculated. They depend on the geometric values, which characterize the size/dimensions of the shell (R, L, δ), of the parameters of wave formation (m, n) and of elastic properties of the material of shell (E, ν).

Let the shell after the break-down of load q under load q_0 accomplish oscillatory motion about basic form. At the torque/moment when bulge stops, energy U_0, communicated by load q in the initial stage of bulge, and work A, produced by load q_0 up to this torque/moment, completely they transfer/convert into the strain energy of shell U.

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Hence is obtained the relationship/ratio

$$U_0 + A = U.$$

If we consider this relationship/ratio as equation for u, then it knowingly has solution with the small q_0 and, on the contrary,

there is not solution with sufficiently large q_0. When there is a solution, shell will reduce its form when it does not exist, bulge, speaking in general terms, unlimitedly it increases. In this case, it is necessary still to bear in mind that u limited by geometric condition (chapter 3)

$$|u| < \frac{\pi R}{2n}.$$

so that the question does not deal with any solution, but about the solution, which satisfies this condition.

The maximum value q_0, at which the solution still exists, is determined by the condition that for it the solution is multiple. Consequently, for this q_0 equation

$$\begin{aligned} U_0 + A - U &= 0, \\ (U_0 + A - U)'_u &= 0 \end{aligned} \qquad (*)$$

they are satisfied simultaneously. If we from these equations exclude u, then we will obtain the relationship/ratio, which links the critical combination of the parameters, that characterize dynamic load q, τ, q_0.

The relationship/ratio indicated between q, τ and q_0 is convenient to present graphically. For this, system of equations (*) is solved relative to q_0 and U_0. Then we will obtain

$$q_0 = \frac{B_1}{C} + \frac{1}{4}\frac{B_2}{C}|u|^{-3/2} + \\ + \frac{3}{4}\frac{B_3}{C}|u|^{-1/2},$$

$$U_0 = \frac{3}{4}B_2|u|^{1/2} + \frac{1}{4}B_3|u|^{3/2}.$$

Now the dependence between values q_0 and U_0 easily is constructed, since q_0 and U_0 are represented depending on one and the same parameter (u).

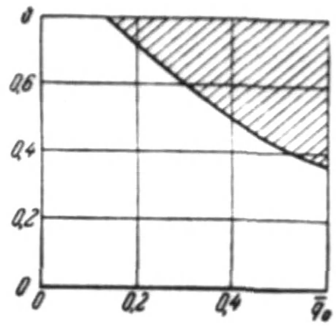

Fig. 52.

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Figure 52 this dependence gives in dimensionless variables $\bar{q}_0$ and ϑ,

$$\bar{q}_0 = \frac{q_0 R}{E\delta}, \quad \bar{q} = \frac{qR}{E\delta},$$

$$\vartheta = 2\bar{q}^2 \left(\mathrm{ch}^2\, \bar{\omega}\tau + \frac{\omega^2}{\bar{\omega}^2}\, \mathrm{sh}^2\, \bar{\omega}\tau\right).$$

Parameters of the wave formation

$$\xi = \frac{Ln}{2\pi Rm} = 1, \quad \eta = \frac{n^2\delta}{R} \simeq 0{,}8.$$

The shaded region answers those values of the parameters of load q, τ, q_0, by which the bulge unlimitedly increases.

2. Critical momentum/impulse/pulse at external pressure. We again examine the hinged suppurted on edges cylindrical shell. Let this shell locate under the action of the stationary external

pressure q_0 and of the short-term (during τ) considerable pressure q. Let pressure q_0 be less than the upper critical value, but q is much more it. Under pressure q the shell loses the stability and it begins to be swelled. It can happen, that the bulge, which appears under pressure q, continues unlimitedly to increase. Task consists of the determination of the critical combination of values q, τ, q_0, by which is realized that indicated possibility. This task does not differ in principle from that examined in p. 1, and therefore our presentation will here be sufficient to short ones.

For sagging/deflections w in the initial stage of bulge, we accept the expression

$$w = u(t)\sin\frac{\pi x}{L}\sin\frac{ny}{R}.$$

Dependence of u on t is determined from stability condition of the functional

$$\int (K - U + A)\,dt,$$

where K - a kinetic energy of motion, U - strain energy of shell, A - the produced by pressure q work by deformation.

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We have

$$K = \frac{1}{4}\pi R L \gamma \delta \dot{u}^{2},$$

$$U = \frac{1}{4}\omega^{2}\pi R L \gamma\, \delta u^{2},$$

where ω - natural vibration frequency, which corresponds to the form of wave formation accepted.

Let us calculate work A. We have

$$A = q\,\Delta V,$$

where ΔV - change in the volume, limited by shell. It is obvious,

$$\Delta V = \int_{0}^{L} \Delta S\, dx,$$

where ΔS - change in the cross-sectional area during the deformation of cylinder, i.e.,

$$\Delta S = \frac{1}{2}\int_{0}^{2\pi} (R^{2} - (R + w)^{2})\, d\varphi, \quad \varphi = \frac{y}{R}.$$

Noting that

$$\int_{0}^{2\pi} w\, d\varphi = 0,$$

we obtain

$$\Delta S = \frac{1}{2R}\int_{0}^{2\pi R} w^{2}\, dy.$$

Consequently,

$$\Delta V = \frac{1}{2R} \int \int w^2 \, dx \, dy = \frac{\pi L}{4} u^2.$$

Thus,

$$A = \frac{\pi L q}{2} u^2.$$

Now it is easy to write the equation of motion of shell in the initial stage of bulge.

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It takes the form

$$u'' - \bar{\omega}^2 u = 0,$$

where

$$-\bar{\omega}^2 = \omega^2 - \frac{q}{R\gamma\delta}.$$

The solution of equation for u, which turns into zero with t=0, will be

$$u = c \operatorname{sh} \bar{\omega} t.$$

For determination constant c we will use the same considerations, as in the examination of axial compression in p. 1. As a result of the loss of stability and bulge, energy U_0, communicated by pressure q to

shell, transfer/converts into kinetic energy of motion. Energy U_0 per the unit surface area is equal to $\delta\sigma^2/E$, where σ - the compression stresses in shell, caused by pressure q, E - modulus of elasticity of material. Taking into account, that $\sigma=qR/\delta$, we find

$$U_0=\frac{2\pi q^2R^3L}{E\delta}.$$

Comparing energy U_0 with kinetic energy of motion at the initial moment, we will obtain

$$\frac{2\pi R^3Lq^2}{E\delta}=\frac{1}{4}\pi RL\gamma\delta u'^2.$$

Hence the speed of bulge at the initial moment is equal to

$$u'=2\sqrt{2}\frac{R}{\delta}\frac{q}{\sqrt{E\gamma}}.$$

Since, on the other hand, $u'=c\bar{\omega}$, then

$$c=2\sqrt{2}\frac{R}{\delta\bar{\omega}}\frac{q}{\sqrt{E\gamma}}.$$

Thus, in initial stage bulge occurs according to the law

$$u(t)=\frac{2\sqrt{2}R}{\bar{\omega}\delta}\frac{q}{\sqrt{E\gamma}}\operatorname{sh}\bar{\omega}t.$$

Up to the torque/moment of the break-down of load q, reported by it to shell the energy will be equal to

$$U_0=\frac{2\pi R^3Lq^2}{E\delta}\left(\operatorname{ch}^2\bar{\omega}\tau+\frac{\omega^2}{\bar{\omega}^2}\operatorname{sh}^2\bar{\omega}\tau\right).$$

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Let us study now the deformation of shell in time with considerable bulge. In chapter 3 given common/general/total expression for the strain energy of shell U and of the produced by pressure q work A with the assigned/prescribed character of sagging/deflections y(x). If the variables x and y are standardized, set/assuming

$$x = \frac{L}{2}\bar{x}, \quad y = \frac{\pi R}{2n}\bar{y}.$$

that

$$U = \frac{\pi^3 E\delta^3}{24(1-\nu^2)n^4}\left(\frac{R}{L}\right)J.$$

$$A = \frac{\pi^3 \bar{q} E\delta^3}{8n^2\varepsilon}\left(\frac{R}{L}\right)\lambda.$$

where

$$J = \frac{\pi^2}{n^2}\left(\frac{R}{L}\right)^2 \int_{-1}^{1} y_{xx}^2\,dx + 2\nu \int_{-1}^{1} y_x^2\,dx +$$

$$+ 6(1-\nu^2)\frac{c}{\sqrt{\varepsilon}} \int_{-1}^{1} |y_{xx}|^{1/2}\,dx + \text{const},$$

$$\lambda = \int_{-1}^{1} y^2\,dx, \quad \varepsilon = \frac{R\delta}{L^2}.$$

In expression J and λ, the feature above standardized/normalized alternating/variable $\bar{x}$ and $\bar{y}$ for simplicity of recording is lowered.

During the investigation of the static equilibrium of shell under this pressure, we approached the plotted function of sagging/deflections $y(x)$ by two rectilinear cuts, smoothly adjoining the parabola in section $|x| < a$. In the course of this investigation, it was explained that value a was small.

If now in the dynamic task of the motion of shell of proceeding in question from the same method of approaching the sagging/deflections, considering it previously a small, then let us arrive at the same expression of strain energy, as in static task. This is connected with the smallness of the parameter $\varepsilon = R\delta/L^2$. The physical cause for this agreement lies in the fact that the strain energy is concentrated in vicinity $|x| < a$, and the basic portion of kinetic energy is concentrated out of this vicinity, that is when $|x| > a$.

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Following the method, presented in Chapter 3, let us find expressions for strain energy and by that produced by external pressure work. They take the form

$$U = A_1 \sqrt{\lambda} + A_2\lambda + A_0, \quad A = Cq_0\lambda,$$

where A_1, ... C - the constants which easily are determined.

If shell after the break-down of load q accomplishes oscillations about basic cylindrical form, then at torque/moment, when the speed of motion is equal to zero ($\lambda'=0$), energy U_0 and work $A(\lambda)$ completely transfer/convert into strain energy $U(\lambda)$. Thus, at this moment

$$U(\lambda)=U_0+A(\lambda). \qquad (*)$$

If we this relationship/ratio consider as equation relatively λ, then it has solution every time that shell reduces its form, accomplishing oscillations, and is not solution, if bulge unlimitedly increases. Peak load q_0, with which equation (*) has solution, is characterized by the fact that the equation relative to λ

$$(U-U_0-A)'_\lambda=0 \qquad (**)$$

has the same solution as equation (*). The critical combination of parameters q, τ, q_0 of dynamic load is determined by the relationship/ratio

$$f(q,\ \tau,\ q_0)=0,$$

which is obtained from equations (*) and (**) by the exception/elimination of the parameter λ.

Let us find relationship/ratio f=0. We have

$$U_0 = A_1\sqrt{\lambda} + A_2\lambda + A_0 - Cq_0\lambda,\ 0 = \frac{1}{2}A_1\frac{1}{\sqrt{\lambda}} + A_2 - Cq_0.$$

determining from the second equation $\sqrt{\lambda}$ and substituting it in the first equation, we will obtain the unknown relationship/ratio.

§ 3. Large oscillations of cylindrical shells.

The fact that the ability of shell to resist external agency usually weakens with considerable bulge, has by its consequence decrease of natural vibration frequency during an increase in the amplitude.

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The more precise investigation of this question will be given in p. 1. The decrease of natural vibration frequency during an increase in the amplitude has, in turn, important consequence. Specifically,, disturbance/perturbation, which is found in resonance with small oscillations, does not lead to the unlimited growth/build-up of the latter. The study of this question is contained in p. 2.

1. Natural oscillations of cylindrical shell. Let the cylindrical shell, hinged supported on edges, accomplish free

oscillations with the same periodicity of sagging/deflections over surface, which possesses function $\sin \frac{\pi x}{L} \sin \frac{ny}{R}$. If oscillations are small, then the normal sagging/deflection of shell is equal to

$$w = u(t) \sin \frac{\pi x}{L} \sin \frac{ny}{R},$$

where to u(t) satisfies the equation

$$u'' + \omega^2 u = 0.$$

Here ω - frequency. It in a specific manner is expressed as the parameter of wave formation n, geometric characteristics and mechanical properties of shell.

Let us examine now the large oscillations of shell. Let us find the equation of these oscillations. It is the equation of Euler for functional

$$\int (K - U)\, dt,$$

where K - kinetic kinetic energy, U - strain energy of shell.

In chapter 3 (page 222) for strain energy is obtained the expression

$$U = \frac{\pi^3 E \delta^3}{24(1-\nu^2) n^2} \left(\frac{R}{L}\right) J + \text{const},$$

$$J = \frac{\pi^2}{n^2} \left(\frac{R}{L}\right)^2 \int_{-1}^{1} y''^2 \, dx + 2\nu \int_{-1}^{1} y'^2 \, dx +$$

$$+ 6(1-\nu^2) \frac{c}{\sqrt{\varepsilon}} \int_{-1}^{1} |y''|^{1/2} \, dx, \quad \varepsilon = \frac{R\delta}{L^2}.$$

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Function $y(x)$ in a known manner is connected with the normal sagging/deflection of shell. So as in the tasks of the static loading of shell (Chapter 3), function $y(x)$ let us approach the function of the form

$$y = \frac{ax^2}{2} - \frac{a\alpha^2}{2} + a\alpha(1-\alpha) \quad \text{при}^{(1)} \quad |x| \leqslant \alpha,$$

$$y = a\alpha(1 - |x|) \quad \text{при} \quad |x| \geqslant \alpha.$$

Key: (1). with.

The graph of this function consists of parabola in section $|x| \leqslant \alpha$ and two smoothly adjacent it rectilinear cuts with ends at points (-1.0) and (+1.0). If we assume a priori that the parameter α is low, then, as in the case of static tasks, energy during the assigned/prescribed common/general/total deformation, limited by the condition

$$\int_{-1}^{1} y^2 \, dx = \lambda = \text{const},$$

is determined from the condition for achievement min J. The reason for this in the fact that under the assumption indicated the strain energy is concentrated in low region $|x|\leqslant a$. while kinetic kinetic energy it is concentrated in essence out of this region. Thus, the determination of energy of deformation in the task in question of the oscillations of shell is reduced to the determination of the minimum of functional J under condition λ=const.

Substituting the value of $y(x)$ in the integrals, entering J, with small α let us have

$$J=\left(\frac{\pi}{n}\right)^2\left(\frac{R}{L}\right)^2 2\alpha a^2+2(\nu-1)2\alpha^2a^2+12(1-\nu^2)\frac{c}{\sqrt{\varepsilon}}\alpha a^{1/2}.$$

Parameter of the deformation

$$\lambda=\frac{2}{3}a^2\alpha^2.$$

Minimum J is located easily, and for it is obtained the expression

$$\min J=J_0=3{,}2\left(\frac{\pi}{n}\right)^{2/3}\left(\frac{R}{L}\right)^{2/3}\frac{1}{\varepsilon^{1/3}}\sqrt{\lambda}+6\nu\lambda.$$

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In view of the fact that the plotted function $y(x)$ is close to the broken line, which consists of two component/links, the deformation of shell can be characterized by the maximum sagging/deflection u(t). Let us establish communication/connection between u and λ. In the standardized/normalized variables we have

$$\lambda=\int_{-1}^{1} y^2\,dx.$$

In initial variables it will be

$$\lambda=\frac{8n^2}{\pi^2R^2L}\int_{-L/2}^{L/2} y^2\,dx.$$

The sagging/deflection of shell is assigned by the function

$$\tilde{y}=\frac{\pi}{2n}\,y.$$

Therefore

$$\lambda=\frac{32n^4}{\pi^4R^2L}\int_{-L/2}^{L/2} \tilde{y}^2\,dx.$$

Since during the deformations of the curve/graphs of function $\tilde{y}(x)$ in question it is close to broken line of two component/links, the

$$\tilde{y}(x)\simeq\frac{2x}{L}\,u(t),$$

where u(t) - maximum sagging/deflection (sagging/deflection with x=0). Hence

$$\lambda=\frac{32n^4}{3\pi^4}\,\frac{u^2}{R^2}.$$

Substituting this value λ in expression J_0, we find the strain energy of shell in dependence on the maximum sagging/deflection u:

$$U = 2A_1'|u| - k'^2u^2 + A_0'.$$

where A'_0, A'_1 and k - constants, moreover

$$A_1' = 1,15n^{4/3}\left(\frac{R}{L}\right)^{2/3}\frac{1}{\varepsilon^{1/3}}\frac{\pi^3E\delta^3}{24(1-\nu^2)L}.$$

Let us find now the kinetic energy K of the motion of shell. Since function $\tilde{y}$, which assigns sagging/deflection, changes actually linearly in each interval (-L/2, 0) (L/2, 0), it turns into zero with x=±L/2 and has value of u(t) with x=0, then

$$K = \frac{1}{2}\int\int \delta\gamma\left(\frac{2xu'}{L}\right)^2 dx\,dy = \frac{\pi RL\,\delta\gamma u'^2}{6}.$$

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Let us compose the equation of the oscillations of shell during large deformations. We have

$$K - U = \frac{\pi RL\,\delta\gamma}{6}\left(u'^2 - 2A_1|u| + k^2u^2\right) + \text{const},$$

where A_1 and k - constants, moreover

$$A_1 = 3,1n^{4/3}\frac{E}{\gamma}\frac{\delta^2}{(\delta R^2)^{1/3}L}.$$

Hence the equation of oscillations with large sagging/deflections

takes the form

$$u'' \pm A_1 - k^2 u = 0,$$

where the sign "+" must be taken with u>0, and sign "-" with u<0. Thus, we have two equations, that describe the motion of the shell:

$$u'' + \omega^2 u = 0 \quad \text{при} \overset{(1)}{\text{малом}} \ |u|,$$
$$u'' \pm A_1 - k^2 u = 0 \quad \text{при} \overset{(2)}{\text{большом}} \ |u|.$$

Key: (1). with small. (2). with large.

Let us introduce the function ϑ(u), determined by the conditions

$$\begin{aligned} &1)\ \vartheta(u) = -\vartheta(-u), \\ &2)\ \vartheta(u) = \omega^2 u \quad \overset{(1)}{\text{при}}\ 0 \leqslant u \leqslant \frac{A_1}{\omega^2 + k^2} = a, \\ &3)\ \vartheta(u) = \pm A - k^2 u \quad \text{при}\ |u| \geqslant a. \end{aligned}$$

Key: (1). when.

Let us examine now the equation

$$u'' + \vartheta(u) = 0. \qquad (*)$$

With small $|u|$ it coincides with equation $u'' + \omega^2 u = 0$, while with large $|u|$ - with equation $u'' \pm A_1 - k^2 u = 0$. We will assume that equation (*) describes the motion of shell in entire range of a change in sagging/deflection u(t). Under this hypothesis let us examine a question concerning the effect of amplitude on the natural vibration

frequency of shell.

First of all, we note that if the amplitude does not exceed a, then frequency is constant and equal to ω. With by an increase in the amplitude the frequency will decrease. Let us find frequency with the amplitude, greater than a.

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Let α - amplitude of oscillations. Let us assume

$$\lambda = \int_0^\alpha \vartheta(u)\,du.$$

Let us multiply equation (*) by u' and will integrate in limits (0, u). We will obtain

$$u'^2 + \int_0^u \vartheta(u)\,du = \text{const}.$$

Integration constant is equal to λ, since u'=0 with u=α. Thus,

$$u'^2 + \int_0^u \vartheta(u)\,du = \lambda.$$

Hence

$$\frac{du}{\left(\lambda - \int_0^u \vartheta(u)\,du\right)^{1/2}} = dt.$$

Integrating this equation within the limits of the fourth of the period, which corresponds to integration for u from 0 to α, we will

obtain the period of oscillations

$$\tau(\alpha)=4\int_0^{\alpha}\frac{du}{\left(\lambda-\int_0^{u}\vartheta(u)\,du\right)^{1/2}}.$$

Consequently, frequency is equal to

$$\omega(\alpha)=\frac{2\pi}{\tau(\alpha)}.$$

Let us estimate natural vibration frequency in the case when amplitude $\alpha \gg a$. In connection with this let us note two facts. First, in the expression

$$\vartheta(u)=\pm A_1-k^2u$$

at geometrically allowed values of $k (\lambda<2/3)$ principal term it is $\pm A_1$, and term/component/addend k^2u has the subordinate value.

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In the second place, with $\alpha \gg a$ the value $\vartheta(u)$ when $\vartheta(u)$ in $|u|<\alpha$ expression $\tau(\alpha)$ little affects value $\tau(\alpha)$. Therefore with $\alpha \gg a$ formula for $\tau(\alpha)$ it is possible to accept $\vartheta(u)=A_1$. Then we obtain

$$\tau(\alpha)=8\sqrt{\frac{\alpha}{A_1}}.$$

Respectively frequency with large amplitude α will be

$$\omega(\alpha)=\frac{\pi}{4}\sqrt{\frac{A_1}{\alpha}}.$$

2. Forced oscillations of shells. Let us examine the forced oscillations of the cylindrical shell, hinged supported on edges. In p. 1, we found the equation of free oscillations. It takes the form

$$u'' + \vartheta(u) = 0,$$

where the function ϑ is determined by the conditions:

$$\begin{array}{lll} 1) & \vartheta(u) = \omega^2 u & (1)\ \text{при}\quad |u| \leqslant a, \\ 2) & \vartheta(u) = \pm A_1 - k^2 u & \text{при}\quad |u| \geqslant a. \end{array}$$

Key: (1). with.

This equation is the equation of Eylera - Lagrange for the functional

$$\int (K - U)\, dt,$$

where K - kinetic kinetic energy, U - strain energy. The equation of forced oscillations will be the equation of eylera - Lagrange for the functional

$$\int (K - U + A)\, dt,$$

where K and U have previous value, A - produced by external disturbance/perturbation work. Hence it follows that the equation of forced oscillations will take the form

$$u'' + \vartheta(u) + \sigma = 0.$$

Let us find function σ for the different methods of disturbance/perturbation. Page 275.

Let the disturbance/perturbation be created by concentrated force of f, normal to the surface of shell,

$$f = f_0 \sin \bar{\omega} t.$$

Let us designate through L' the distance of the point of the application of force f from the nearest edge of shell ($L'<L/2$). In view of the fact that the sagging/deflection of shell on any half generatrix changes almost linearly and is equal to zero at the edges of shell, sagging/deflection at the point of application of force will be $\simeq 2u(t)L'/L$. Hence for work A, is obtained the expression

$$A = f_0 \sin \bar{\omega} t \, \frac{2L'}{L} \, u(t).$$

For the kinetic energy K, we will obtain the expression of the form

$$K = K_0 u'^2.$$

Hence it follows that the function σ in the equation of oscillations will take the following form:

$$\sigma = \frac{2 f_0 L'}{K_0 L} \sin \bar{\omega} t = \lambda \sin \bar{\omega} t.$$

Thus, the equation of forced oscillations in the case in question during large oscillations will be

$$u'' + \vartheta(u) + \lambda \sin \bar{\omega} t = 0.$$

Let now the disturbance/perturbation be created by the fluctuating, evenly distributed on shell pressure q, which are changed according to the law

$$q(t) = q_0 \sin \bar{\omega} t.$$

Produced by this pressure work of the deformation of shell will be

$$A = q\,\Delta V,$$

where ΔV - change in the volume, limited by shell, during deformation. Value ΔV was determined in Chapter 3 during the study of the static loading of shell by external pressure.

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If we use the obtained there result, then let us find

$$\Delta V = cu^2,$$

where c - certain constant, which depends on the geometric dimensions

of shell and parameter of wave formation n. Consequently, work A in the load case in question will have the expression

$$A = c'qu^2 \sin \bar{\omega} t.$$

Respectively the equation of oscillations with large sagging/deflections will be

$$u'' + \vartheta(u) + \lambda u \sin \bar{\omega} t = 0.$$

In the case when disturbance/perturbation is created by the evenly distributed along the edge of shell fluctuating pressure

$$p = p_0 \sin \bar{\omega} t.$$

equation is the same, but with its value of coefficient λ.

Let us assume now that on shell affects the disturbance/perturbation of sufficiently small intensity, but with frequency $\bar{\omega}=\omega$. If the natural vibration frequency of shell did not change during an increase in the amplitude, then this disturbance/perturbation, no matter how was small its intensity, it would lead to unlimited oscillation buildup (to unlimited increase in the amplitude). However, as shown in p. 2, frequency retains constant value (ω) to those pores, while amplitude does not exceed value

$$a = \frac{A_1}{\omega^2 + k^2}.$$

With the larger amplitude of oscillations, the resonance of disturbance/perturbation and natural oscillations is disrupted. Hence we make important the conclusion that the disturbance/perturbation of sufficiently small intensity with frequency ω swings shell to the amplitude

$$\alpha < \frac{A_1}{\omega^2}.$$

Further oscillation buildup at the excitation frequency ω is possible only because of an increase in the intensity of disturbance/perturbation.

In conclusion we want to make the following observation. In our all examinations we assumed the unlimited elasticity of the material of shell.

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To the real shells which possess the limited elasticity, our conclusion/derivations are used only during satisfaction of the specified conditions. Specifically,, the voltage/stresses in the material of the shells, which appear during large deformations, determined on the appropriate formulas from Chapter 3, must not exceed elastic limit. In view of the fact that the character of deformations during oscillations the same as and during static

deformations with bulge, the limitation indicated takes the same form, as for a back, examined in Chapter 3. Speaking in general terms, it is reduced to the fact that the shell must be sufficient fine/thin.

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SUPPLEMENT II ISOMETRIC TRANSFORMATIONS OF CYLINDRICAL SURFACES.

According to principle A, the investigation of the supercritical elastic states of cylindrical shell is reduced to the examination of certain functional, determined during the isometric transformations of basic form, during the satisfaction of some boundary conditions, which correspond to the method of supporting the shell. For the isometric transformations, constructed in chapter 3, these boundary conditions are not satisfied.

True, when supercritical deformation differs in terms of the periodicity of structure along the length of shell, for example during axial compression either with combined loading with prevalence axial compression or combined loading with the predominance of axial compression, the role of boundary conditions considerably descends. The satisfaction of boundary conditions in the examination of the corresponding tasks hardly can substantially influence the results Chapter 3. It is a different matter when supercritical deformation is accompanied by the education/formation of continuous bulges to entire

length of shell, for example, in the case of external pressure and twisting. Here, apparently, it is not possible to disregard boundary conditions. In connection with this in present supplement, keeping in mind the appropriate application/appendices, we will examine the isometric transformations of cylindrical surface under the condition for hinged support along edge. The obtained results can have an application during the study of supercritical ones by the deformation of the closed cylindrical shells under external pressure and during twisting, and also in the examination of the supercritical elastic states of cylindrical panels in all versions of loading.

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1. Some common/general/total properties of developable surfaces. We examine developable surfaces with disturbance of regularity (twofold differentiability) along individual lines. On these lines it can occur either the disturbance/breakdown of smoothness (formation of fin/edges), or breakage for the normal curvatures which, however, are assumed to be those limited. According to definition, developable surface allow/assumes imposition on plane with the preservation/retention/maintaining of the lengths of curves, therefore, and the angles between them. Let us examine some properties of developable surfaces, utilized subsequently.

Let us, first of all, note that each smooth point of developable surface is the internal point of the rectilinear cut, which lies by pillar on surface (linear generator). For the regular, twice differentiated surfaces, this is well known fact from differential geometry. For the more total surfaces which we examine that actually is established in work [13]. Really/actually, since point is smooth, and the smoothness of surface is disrupted along individual lines, then in this point smooth vicinity. In view of the fact that the normal curvatures wherever they exist, are limited, the vicinity in question is the surface of the limited external curvature. while each point of this developable surface is the internal point of cut, the pillar of that lying on surface.

If through the point of developable surface pass two linear generator, then this point has flat/plane vicinity, that is the vicinity, which is the piece of plane. Really/actually, sufficient to show that any cut AB with ends on generatrices belongs to surface (Fig. 53). Let us connect points A and B shortest γ on surface. Let us expand/scan now the vicinity of point O to plane. In this case, cuts OA and OB linear generator will pass in the rectilinear cuts of the same length and with the same angle between them. Consequently, in this case process/operations will not change and the distance between points A, B.

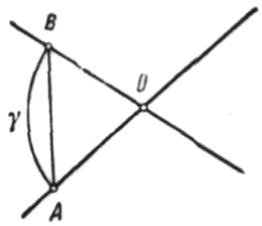

Fig. 53.

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Shortest γ will pass into the shortest, that is, in the rectilinear cut, which connects points A, B. We hence consist that the distance in space between points A and B is equal to length curved γ. But this can be only in such a case, when curved γ coincides with the rectilinear cut AB. Hence it follows that cut AB lie/rests on surface.

Let AB represent the linear generator on developable surface. Let us show that if any point of generatrix has flat/plane vicinity, then each internal point of generatrix also has such vicinity (in this case we let us speak, that along generatrix occurs the flattening of surface). Really/actually, let point C of generatrix have flat/plane vicinity. Let us conduct through C the rectilinear cut DE on surface (point C internal for cut DE) so that it would not

lean on AB (Fig. 54). This is possible, since point C has flat/plane vicinity. Let us examine the geodetic quadrangle, limited shortest AD, DB, BE and EA. By the given above reasoning easily it is established that this quadrangle flat/plane, and each internal point of cut AB is internal for a quadrangle, Q. E. D. In exactly the same manner it is proven, that if linear generator AB and CD developable surfaces have the common point (ends they are not eliminated), then geodetic quadrangle with apex/vertexes A, B, C, D (degenerating into triangle, if common point it is the end of generatrix) it will be flat/plane.

If along linear generator developable surface there is no flattening, then it rests by its ends either into fin/edge or into edge of surface. Really/actually, let point A - end linear generator g - be the internal point of surface and does not belong to fin/edge. Then at point A surface is smooth; therefore through point A, is passed the rectilinear cut 6, which lies on surface, moreover point A is its internal point.

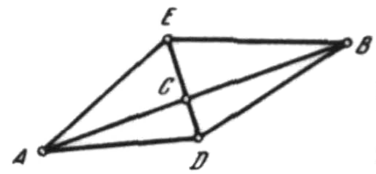

Fig. 54.

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If this cut on the one hand leans on forming g, then A is not end by forming g in spite of assumption. But if cut δ does not lean on forming g, then on that demonstrated above along generatrix occurs flattening, which is also excluded. Affirmation is proved.

If the geodetic curvature of fin/edge γ on developable surface is different from zero, then this fin/edge cannot have flat/plane half-neighborhood. Let us allow contrary. Let one of the half-neighborhoods of fin/edge γ is flat/plane (Fig. 55). Let us assume that this is half-neighborhood from the side of convexity (ω_1). Let us take for γ the arbitrary point A. Tangential plane from surface at point A from the side of half-neighborhood ω_1 coincides with the osculating plane of fin/edge γ. Let us show that the tangential plane of surface from the side ω_2 coincides with the same plane. Really/actually, since surface is run up/turned to plane, then

the geodetic curvatures of fin/edge γ from the side ω_1 and ω_2 are identical and are characterized by only sign. The angles which composes osculating plane of fin/edge with the tangential planes of surface, are determined only by the curvature of fin/edge and by its geodetic curvatures; consequently, these angles are equal. Since one of the tangential planes of surface (from the side ω_1) coincides with osculating plane, then also another tangential plane (from the side ω_2) must possess the same property. As a result it is obtained, that point A is the smooth point of surface in spite of assumption. Affirmation is proved.

2. Qualitative study of isometric transformation of cylindrical surface. Let the cylindrical surface undergo gecmetric bending with the formation of the system of the congruent dents to entire length of initial surface, correctly arrange/located in circular direction (Fig. 56a). Assumption about the fact that the dents apply to entire length of surface, we understand in the sense that its limiting fin/edges γ, γ' connect the points of basis/base A, B.

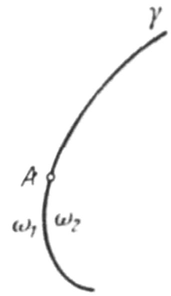

Fig. 55.

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The regular arrangement of dents means that the deformed surface has periodic structure in circular direction. Each dent we will assume convex and symmetrical relative to the plane, passing through its end-points A and B on the basis of cylinder. The task, which we want to solve, consists in the qualitative description of the form of the deformed surface. In view of the periodicity of structure, it is possible to be restricted to the examination of the part of the surface, arrange/located between two adjacent planes of symmetry α and α'. Plane α' passes along the axis of dent, while plane α - in the middle between two dents (Fig. 56b).

First of all, we will note that the lines of intersection l and l' surface with the planes of symmetry α and α' are geodetic. On initial cylindrical surface to them correspond linear generator. This

follows from the fact that the planes α and α' are the planes of symmetry, therefore, they intersect surface orthogonally.

Let us assume that on line l' there are no rectilinear cuts. Then through each point P of this line passes linear generator, which rests by end into fin/edge γ. Really/actually, linear generator cannot go in direction l', since in this case it on it leans according to the property of geodetic ones, and by hypothesis l', are not contained rectilinear cuts. It remains to assume that forming, passing through point P, it is not perpendicular to plane α' and does not lie/rest at this plane. But then on symmetry relative to plane α' there is one additional generatrix, passing through P and, therefore, P has flat/plane vicinity. But this so is impossible due to the absence of straight portions curved l'. Thus, through each point P by curve l' pass linear generator, perpendicular planes α'. This generatrix rests by its end into the fin/edge of surface γ (p. 1). Hence it follows that the surface in question in the region of dent is cylindrical, with generatrices, perpendicular to the plane of symmetry α'.

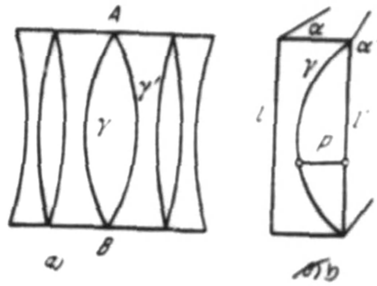

Fig. 56.

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Let us explain now how is arranged our surface out of dent, that is, in the region between the fin/edge γ and curve l. In contrast to the case curve l' for a curve l it is not possible to assume the absence of straight portions. This assumption immediately leads us to conclusion about the fact that the surface in question and in the region between l and γ is also cylindrical, with generatrices perpendicular ones plane α. But for this surface (this the constructed in chapter 3 surface Z) support condition for edges on circumference in an obvious manner is not satisfied. Thus, to curved l must be straight portions.

We confirm that at least one of the ends of the rectilinear cut curved l belongs to edge of surface. Really/actually let us assume that both of ends A and B of the straight portion δ of line l are its

internal points. Since points A and B do not knowingly have flat/plane vicinities, then through them pass linear generator g_A and g_B, the perpendicular planes α, abut against fin/edge γ. The part of the surface ω, limited by generatrices g_A, g_B and curves γ, l, is flat/plane. Really/actually, since curve γ is directed by convexity to side ω, then through each internal point P of this region passes geodetic, not intersecting γ, with ends either in cut δ or on generatrices g_A and g_B. By the reasoning, similar to reasoning from p. 1, it is easily establish/installed, what this geodetic is rectilinear cut. Hence it follows that the region ω is flat/plane. Since ω flat/plane region, then in the section between generatrices g_A and g_B fin/edge γ has flat/plane half-neighborhood, which is excluded in view of convexity γ (p. 1). Affirmation is proved.

Further, we confirm that near ends the curve l must be definitely rectilinear. Really/actually, otherwise will be located the sequence of points Q to curved l, that converges toward the end, through which pass linear generator, perpendicular planes α, abut against fin/edge γ. Transfer/converting to limit, we consist that through the end l passes linear generator, which goes along edge of surface. But this is impossible, since edge of surface does not contain straight portions (circumference). As a result we come to the following two possibilities.

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Either curve l is rectilinear cut or it contains exactly two rectilinear cuts, each of which by one end abut against edge of surface. Let us examine each of these possibilities.

Let us assume that the surface in question has a plane of symmetry β, parallel to basis/bases and situated in the middle between them. Let us examine the arc AB of the curve of intersection of the plane of symmetry β with surface. We confirm that if the curve AB contains rectilinear cut, then one of its ends is point B. Really/actually, let us allow contrary. Then on arc AB is located the rectilinear cut A'B' (Fig. 57a), B'≠B, and A' can coincide with A. Since points A' and B' do not knowingly have flat/plane vicinities, then through them pass linear generator, perpendicular planes β and abut against edge of surface. After this by known method we consist that the part of the surface, limited by these generatrices, flat/plane. Consequently, edge of surface contains rectilinear cut. But this is impossible.

Let us show that arc AB definitely has straight portion A'B. Really/actually, otherwise through each point of arc AB, pass linear generator, perpendicular planes β, with ends at edge of surface. This generatrix lie/rests on initial cylindrical surface. Let us examine

the line AC of the intersection of plane β with surface. It consists of circumference AB and rectilinear cut BC. On isometry on initial cylindrical surface to it corresponds circular section/cut and, therefore, arc length AC is equal to the arc length of the circumference of edge. Moreover, it is known to be shorter than this arc due to the straightness of section BC. Thus, to the curved AB is, and besides one, rectilinear cut with end at point B.

Let us turn to two mentioned above possibilities for a curve l. Let us begin from the case when l - rectilinear cut.

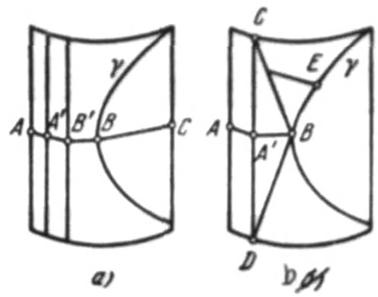

Fig. 57.

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On that demonstrated, on AB there is the rectilinear cut A'B. Through each point of arc AB', pass linear generator, perpendicular planes β, with ends at edge of surface. The part of the surface, carrier these generatrices, coincides with initial cylindrical surface.

Let us explain the structure of the surface between forming CD and by fin/edge γ (Fig. 57b). Since A'B and CD rectilinear cuts on surface, then is triangle BCD flat/plane (p. 1). Let us examine the region between generatrix BC and fin/edge γ. We confirm that this region does not contain flattenings and each of its generatrices rests by one end into edge of surface, but by others - into fin/edge γ. Really/actually, if rectilinear generatrix by one end abut against BC, and by another into fin/edge γ at certain point E, then arc BE of fin/edge γ has flat/plane half-neighborhood, which is impossible (p.

1). But if it rests by one end into edge, by others - in BC, then at edge of surface there will be rectilinear cut, which is also impossible. Both of ends of generatrix neither edge nor fin/edge simultaneously can belong. Thus, one end of generatrix at edge of surface, and another - on fin/edge γ.

Let us allow now that at certain point of the region in question is a flattening (flat/plane vicinity). Let us conduct two linear generator through the points of this vicinity. Then either edge of surface or fin/edge γ these generatrices intersect at different points. If this there will be fin/edge, then its arc between points of intersection has flat/plane half-neighborhood. If this there will be edge, then of cuttings off between points of intersection it must be rectilinear.; however, even that, and another is eliminated. Thus, if the section/cut of surface by plane α is rectilinear cut, then surface has structure, indicated in Fig. 58a, where the shading designated the directions of generatrices, and the nonshaded sections - flat/plane.

Let us examine now the second case: to curved l are two straight portions, arranged/located are symmetrical relative to plane β and the abut against edge of surface.

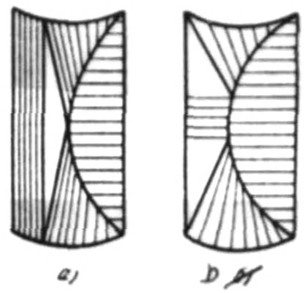

Fig. 58.

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Through each point as curve l between rectilinear cuts passes linear generator, perpendicular planes β, abut against fin/edge γ. The structure of surface on the remaining part is explained by the examination, similar preceding/previous. Final result is represented in Fig. 58b, where, as in the preceding case, by shading is marked the direction of generatrices, but the unhatched parts - flat/plane.

We examined the isometric transformation of cylindrical surface under the condition of the axial symmetry of dents. This transformation corresponds to the symmetric loading of shell, for example, by uniform external pressure. In other load cases, for example, during twisting, isometric transformation, possessing the periodicity of structure in circular direction, has centrally symmetric dents. This transformation can be qualitatively

investigated by analogous method. We will not give this investigation and will formulate only final result, assuming surface in the region of the dent of cylindrical. This result is represented in Fig. 59 and does not need explanations.

3. Analytical description of isometric transformation. The practical use of isometric transformation of cylindrical surface, obtained in p. 2, for study of the supercritical elastic state of cylindrical shell assumes the convenient analytical representation of this transformation. Now we will give this idea for that case when dents have an axis of symmetry and it is sufficiently elongated along the length of shell.

Let us introduce during the scan/development of the cylinder of coordinate t and β, after accepting for axle/axis t the axis of the symmetry of the predicted dent, and for axle/axis β, - an edge of surface. Let in these coordinates the predicted fin/edge γ be assigned by the equation

$$\beta=\beta(t).$$

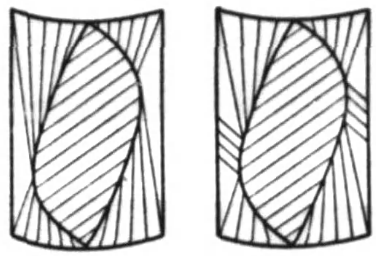

Fig. 59.

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Let us introduce, further, the three-dimensional/space rectangular coordinates X, y, z, connected with the surface in question, after accepting for x axis straight line, which connects the end-points of dent at edge of surface, and for y axis tangent to edge of surface in one of these points (Fig. 60).

The equations of fin/edge γ (boundary of dent) in these coordinates will be written as follows:

$$x = x(t), \quad y = \beta(t), \quad z = \lambda(t).$$

Being limited to the practically important case when λ' not too greatly, it is possible to accept $x(t) \simeq t$. Then the equations of fin/edge take the form

$$x = t, \quad y = \beta(t), \quad z = \lambda(t).$$

The analytical description of isometric transformation in the

version, presented in Fig. 58b, assumes the determination of function $\lambda(t)$ and functions $\xi(t)$, by assigning y - coordinate of the end linear generator depending on parameter t (see Fig. 60). For these two functions can be two differential equation, which express by itself the geometric condition of the applicability of the surface in question for plane. Let us find these equations.

Since surface is isometric plane, then along fin/edge the tangential planes of surface form with the osculating plane of fin/edge equal angles. Let us conduct from point (t) on fin/edge γ the osculating plane of fin/edge and the tangential planes of surface. Normals to them lie/rest at one plane and in view of the character of the deformations in question are formed small angles. Therefore the indicated condition of equality the angles between normals can be attributed to their projections on plane x=0. During planning to this plane of the standard of tangential plane from the side of dent, it (standard) will pass into z-axis. Let us find the angles, formed by the projections of the standards of other two planes with z-axis.

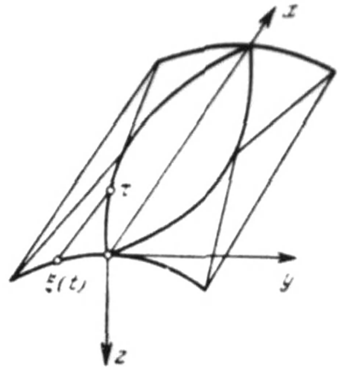

Fig. 60.

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The components of the standard osculating plane of fin/edge along the axes y and z are respectively equal to

$$\begin{vmatrix} \lambda' & 1 \\ \lambda'' & 0 \end{vmatrix} = -\lambda'', \quad \begin{vmatrix} 1 & \beta' \\ 0 & \beta'' \end{vmatrix} = \beta''.$$

The projected angle of standard on plane x=0 with y-axis can be considered equal to $\simeq -\lambda''/\beta''$.

The tangential plane of surface in fin/edge from the side, external with respect to dent, passes through the tangent to fin/edge and it concerns edge of surface. Along edge of surface

$$y = \xi(t), \quad z \simeq \frac{1}{2R}\xi^2(t),$$

where R - radius of basis/base. Therefore the components of the standard of tangential plane along the axes y and z will respectively

be

$$\begin{vmatrix} \lambda' & 1 \\ \frac{\xi\xi'}{R}, & 0 \end{vmatrix} = -\frac{\xi\xi'}{R}, \quad \begin{vmatrix} 1 & \beta' \\ 0 & \xi' \end{vmatrix} = \xi'.$$

The projected angle of standard on plane x=0 with y axis can be considered equal to $\simeq -\xi/R$.

Now the mentioned above condition of the unwinding of surface along fin/edge can be expressed in the form of the following equation:

$$\frac{\lambda''}{\beta''} = \frac{\xi}{2R}. \qquad (*)$$

The condition of applicability for the plane of surface out of dent is reduced to coplanarity by three straight lines: by rectilinear generatrix $(t, \xi(t))$, tangent to fin/edge γ, and by tangential to edge of surface at end forming. This condition lies in the fact that

$$\begin{vmatrix} 1 & \beta' & \lambda' \\ 0 & 1 & \frac{\xi}{R} \\ t, & \beta-\xi, & \lambda-\frac{\xi^2}{2R} \end{vmatrix} = 0,$$

that is,

$$(\lambda - t\lambda') + \frac{\xi}{R}(\beta' t - \beta) + \frac{\xi^2}{R} = 0. \qquad (**)$$

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Differentiating equations (**) and eliminating λ'' with the aid

of equation (*), we will obtain $\xi/2(\beta''t) + \xi'(\beta't-\beta)+\xi\xi'=0$. After multiplication by ξ this equation, is integrated and is obtained $2/3\xi^3+\xi^2(\beta't-\beta)=\text{const}$. Integration constant is equal to zero, since $\xi=0$ with t=0. As a result for $\xi(t)$ is obtained the following expression: $\xi=3/2(\beta-t\beta')$.

This formula can be interpreted geometrically. Specifically,, tangent by curve γ during the scan/development of cylinder intercept/detaches at the edge of the cylinder of cuttings off $\beta-\beta't$, but forming, which proceeds from the same point, intercept/detaches one and a half times larger cut. It is hence not difficult to conclude that the isometric transformation in the version in question will occur to those pores, while the width of dent does not exceed two thirds between the axle/axes of adjacent dents.

Now, when function $\xi(t)$ is known, function $\lambda(t)$ is found with the aid of quadratures from the equation

$$\lambda''=\frac{3\beta''}{4R}(\beta-\beta't).$$

One of these quadratures is fulfilled in the general case (with any $\beta(t)$). Really/actually, after multiplying equation by t and integrating, we will obtain $\lambda't-\lambda=-3/8R(\beta't-\beta)^2$ (integration constant it is equal to zero, since with t=0 it will be $\beta=0$ and $\lambda=0$). Finally function $\lambda(t)$ is represented in the form

$$\lambda=-\frac{3t}{8R}\int_0^t\left(\beta'-\frac{\beta}{t}\right)^2 dt+ct.$$

Integration constant is determined from condition $\lambda(L)=0$, where L - length of cylinder.

We examined the case when adjacent dents were divided by the piece of initial cylindrical surface. In other version when between dents appears cylindrical surface, is perpendicular to the plane of symmetry β (see Section 2), the description of form in this part in no way differs from that which was carried out in chapter 3.

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Epilogue to A. V. Pogorelov's book "geometric stability theory of shells".

I. I. Vorovich.

The problem of elastic stability occupies one of the central places in modern mechanics. Its solution is exclusively important for practice in connection with the wide use of thin-walled cell/elements in technology, especially in its newest branches. At the same time a characteristic feature of problem is its complexity from the point of view of strict mathematical analysis and numerical resolution.

In recent years it is possible to note serious shift/shears in its solution, however, difficulties are so great that in spite of the large effort/forces, which apply here, many fundamental questions are not yet clear, but the numerical results of solving one and the same specific problems in the different authors are obtained with large spread.

For understanding of the reasons, which give rise to the complexity of the task of stability, let us attempt to establish that must be obtained as a result of its solution.

During the operation of the objects, which contain thin-walled cell/elements, in a number of cases these cell/elements can, generally speaking, under the fixed/recorded conditions have several forms of equilibrium and even several stable forms. Of them usually only one is desirable, provided for by designer. Transition into other forms, as a rule, will draw the dangerous phenomena up to structural failure.

Thus, during the solution of a question of stability we should describe the conditions, which ensure the stay of thin-walled cell/element in the desirable, provided for by designer form of equilibrium.

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In the instruments of automation, we frequently encounter the opposite situation: it is necessary to explain the conditions, by which will be provided the transition of one form of equilibrium into another. Consequently, in the stability theory of thin-walled cell/elements must be developed the qualitative methods of determination of a number of forms of the equilibrium of system under these conditions and the methods of evaluating the degree of the validity of each of these forms. These two problems compose the basic content of the stability theory of thin-walled cell/elements. At

present great development underwent the first problem. As far as problem is concerned second, in spite of its importance it began to be develop/processed only recently during the basis of probabilistic methods.

During the analysis of the first problem in essence, they try to explain the limits of a change in the parameters of the loads with which this system has the only form of equilibrium. L. Euler based on the example of buckling indicated the way of finding these limits the basis of transition to the linearized task. This method subsequently began widely to be utilized and it was strictly substantiated. It completely justified itself in connection with rods, frames, plates. To sizable degree contributing to this was the fact that the linearization in these cases is produced in the vicinity of the zero moment stressed state, which can be here designed by comparatively simple, and sometimes also simply elementary means.

However, the attempts to utilize a linearization for solving the tasks of the stability of shells proved to be unsuccessful. Linearized concept in this form, in which it was applied earlier, gave the completely distorted representation of the critical values of loads. It turned out that it one should utilize, linearizing task of region of previously unknown solution or necessary generally to forego linearization and to pass to the direct global investigation

of the nonlinear equations, which describe the deformation of shell. The latter are the complex system of equations in partial derivatives, which contains the parameter on a certain nonlinear boundary-value problem. In a number of cases, this investigation succeeds in conducting strictly; however, the concrete/specific/actual calculation of the characteristic points of the spectrum is necessary to always produce approximately, mainly with direct methods.

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Hence becomes clear the exceptional difficulty of the mathematical study of stability problem of thin-walled cell/elements, about which the question was above. The complexity of task considerably grow/rises, if the material of thin-walled cell/element begins to work beyond elastic limits. Essential difficulties we experience/test even when we attempt to use in the described problem approximation methods. The fact is that median surface of film with loss of stability takes the form which has sections of steady change and sections of a very strong change in the relief. This very complicates the use of direct methods, since this form of median surface it is very difficult to approximate by simple analytical means. For obtaining sufficient accuracy/precision, it is necessary to resort to the use of direct methods in high approach/approximations, which is

extremely cumbersome. The noted above complex character of the state of strain of median surface in large measure is explained by the geometry of its nondeformed state. Here we naturally matched up and to the ideas of the outstanding Soviet geo*meter of A. V. Pogorelov, who developed the method of the analysis of stability of shells, based on the fine/thin analysis of shape factors. In connection with this it is necessary to say that there is a very close connection between the tasks of the theory of shells and by some questions of the theory of surfaces. Widely it is known that the equations of the zero moment stressed state of shell are identical to equations, by which are described infinitesimal bending of its median surface. This fact is one of the manifestations of deep communication/connections between static and geometric relationship/ratios, expressed in static-geometric analogy. Generally, some tasks of the theory of surfaces can be treated as the tasks of the equilibrium of the two-dimensional media, allotted the specific physical properties. So, the task of the unique determination of surface under given conditions of attachment can be treated as the task of a number of forms of the equilibrium of the two-dimensional continuum, which has zero rigidity to bending and infinite rigidity to elongation. It is possible to enlarge the formulation of the problem, after assuming that the continuum has final rigidity to elongation and the energy, accumulated with elongation, is proportional to a change in the area.

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We come, thus, to plateaus problem, and instead of a question concerning the unique determination of this surface it is here expedient to be interested in a number of forms of equilibrium. It is possible to still enlarge the formulation of the problem, after supposing that the strain energy of two-dimensional contiuum is a certain function of a change in the coefficients of the first and second quadratic forms. After posing the problem of finding of a number of forms of the equilibrium of this continuum, we actually come to stability problem of shells, as it are understood in the theory of elasticity. Distinctive features of the constructions of A. V. Pogorelova consists in the fact that here the geometric methods are utilized as the concrete/specific/actual means of the solution of stability problem.

Taking into account a very small change in the lengths on median surface during the real deformations of shell, even if its form in this case substantially changes, A. V. Pogorelov comes to the conclusion that median surface of shell must be close to the isometric representations of its nondeformed state. Since in the practically important cases of the condition of attachment they are such, that the smooth isometric representations are absent, then we come to the need for searching for deformed median surface of shell

in the class of the smooth surfaces, close to the irregular isometric representations of initial form. By these, in particular, is explained the peculiarity of the form of median surface of shell, which it obtains in supercritical stage and about which the question was above. It is important to note that this fact is confirmed on experiments with very films. The fact indicated prompts the method of the approximation of median surface during the use of direct methods. A. V. Pogorelov approximates by its smooth surface, close to certain irregular bending of the initial form of median surface. Appearing in this case arbitrariness the author finds from Lagrange's variation principle. On this basis are investigated the supercritical states of strictly convex hulls with different loads. Under the effect of concentrated force, is establish/installed the absence of "cotton/knock", which is confirmed by the solution this same of task by Bubnov - Galerkin's method in high approach/approximations. For the case of uniform load in this way is obtained the falling/incident part of the characteristic of loading.

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Analogous approach is utilized also for the calculation of lower critical numbers of cylindrical shells with different loads.

Geometric considerations are utilized in the book also for the

calculation of upper critical forces with the only difference that the here for the approximation of deformed median surface are applied its infinitesimal bending. In this case, the author constructs the disruptive fields infinitesimal bending, conjugate/combined in a specific manner along certain line on the basis of electrical installations.

Geometric approximation is the basis of the proposed to A. V. Pogorelov method of solving the dynamic tasks. With the use of Ostrogradskiy - Hamilton's principle, examines the task of dynamic "cotton/knock" for the spherical cupola, loaded by uniform load. We seek the value of the supplementary momentum/impulse/pulse which in state will move shell into supercritical form. Here are examined dynamic tasks for a cylindrical shell.

The powerful aspects of the method of A. V. Pogorelov consist in his clarity, simplicity of the obtained solutions, generality of the formulation of the problems. It is very important to note that in a number of cases A. V. Pogorelov's final formulas obtained careful test work. Very setting of experiments, method of manufacturing the precessional shells, developed by A. V. Pogorelov, must be considered as the significant contribution to the technology of experimentation with shells.

Like any approximation method, method of A. V. Pogorelov has the specific boundaries of their application/use, until now, in many respects unclear. These boundaries are determined by the fact that in the discussed method the form of the approximation of median surface contains only one varied parameter, because of which is satisfied approximately the variation equation of Lagrange or Ostrogradskiy - Hamilton. By force this method it possesses weak sensitivity to the character of external load. Furthermore, method actually considers only one of boundary conditions $\dot{W}|_{\Gamma}=0$, its weak sensitivity to the method of framing.

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For example, conclusion/derivation about the absence of "cotton/knock" in the rigidly attached spherical cupola, loaded by concentrated force, can not confirm itself for the hinged supported cupola. Let us note finally and this fact. Although the approach/approximations, given by geometric method, in a number of cases prove to be completely satisfactory, nevertheless very frequently can get up a question concerning their refinement. Here, in our opinion, within the framework of geometric approach available very difficult prospects.

A question concerning the limits of the applicability of

geometric method is very important. Now thus far from most common/general/total concepts it is possible to assume that the obtained here approach/approximation will be good for very fine/thin and not very slightly curved shells. In connection with this it would be very interesting to obtain the basic results of geometric method via the analysis of the common/general/total equations of the nonlinear theory of shells, accomplishing in these equations these or other passages to the limit.

Certain regret is necessary to express in connection with the fact that the author nowhere compares his results with the results of earlier investigations, although many of the tasks, examined by geometric method, were solved by other previously methods.

At the same time A. V. Pogorelov's book is written clearly and intelligible. It contains necessary for understanding of basic questions information both of the theory of shells and from geometry. The book is interesting for the wide circle of the readers: the specialists in the region of the theory of the elasticity, engineers, who carry out use and calculation of shells, finally for pure/clean mathematicians, who are interested in application/appendices.

The appearance of A. V. Pogorelov's book will unconditionally generate great interest of the specialists in the analysis of the

possibilities which are open/disclosed in the theory of shells with the use of geometric methods.

Work on comprehension and development of these possibilities, regarding the limits of the applicability of gecmetric method no doubt positively will affect the development of stability problem.